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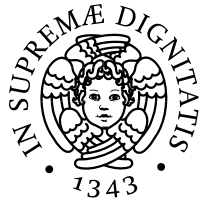

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Current correlators for multi-mode entanglement detection

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Introduction

Entanglement is one of the most interesting features of quantum mechanics; yet, in the last century, it was often considered a mysterious oddity of the theory. Entanglement is indeed the signature of the "non locality" of quantum mechanical theory [1]. Assume, for instance, that we have two spatially separated quantum physical systems. It is reasonable to require that physical transformations and measurements may act on one of the two systems only, i.e. only *local* operations are physically allowed. However, when we perform a local operation on one of the two systems, even the state characterizing the other one may be influenced somehow, surprisingly without any direct mechanical interaction or signaling. When this phenomenon happens, we talk about quantum correlation: the two systems manifest an entangled quantum state.

It was discovered that this particular kind of correlation overcomes any classical system capability of sharing information; this suggested to consider entanglement as a potential asset, on which one could elaborate, for example, new techniques of computation and new methods of communication.

In the last decades several innovative theories, and new matters of study were developed around this concept. Especially in *quantum computation* and *quantum information* entanglement played a main role. In these contexts entanglement started being literally treated like a resource, an irreplaceable ingredient which must be spent in order to implement efficiently the new techniques. It is the essential element that is required to run such quantum based protocols, elaborated in the last fifty years, able to realize unusual effects, like teleportation, safer cryptography, superdense data compression, and so on.

Now, since entanglement has been regarded as a resource, it is clear why we are presently looking for physical systems suitable for our various purposes: preparation, transmission, quantification, and utilization of quantum entanglement.

A class of physical systems which have been studied with fair interest around this matter, is quantum optics. Photon systems show long coherence, in space and time, and they are easy to manipulate and to measure; for these, and other reasons, quantum optical systems have been considered good candidates to become the physical components for the implementation of quantum information protocols.

Only in recent times research on entanglement turned its attention on *condensed matter systems* as well. Indeed, physical science succeeded in developing a mature solid state technology, lately. Moreover, several important theories on solid state systems were elaborated and experimentally verified. In particular, electron transport properties in mesoscopic condensed matter structures have been studied with much interest.

Electron conduction in particular is actually viewed as another potential candidate as experimental context for our quantum informational approach. If we find some degrees of freedom, for the electronic conduction process in our solid sample, uncoupling from the transport dynamic, then we can regard this particular set of degrees of freedom (also known as channels) like a quantum information workspace. Spin, transverse modes, edge states in quantum Hall regime, all these degrees of freedom are acceptable channels for our purposes.

Now, as long as we study conduction properties of electrons in nanostructured solid systems, we can assume the transport process being *coherent*. This leads to two important facts: not only the transmission of quantum states in this context becomes a natural issue, but also we are ensured that transport processes preserve and do not degrade entanglement. However, the production itself of entanglement, its quantification by physical measurements, and controlled manipulation of quantum states in such systems, are problems still open to physical study.

In these years various proposals of a solution for these questions have been formulated; see for example refs. [2] [3] [4] [5] [6] [7] [8]. In this thesis we are going to present a theoretical operative scheme for detection and lower-bound estimation of multi-channel entanglement held by transport electron states in multi-terminal mesoscopic condensed matter systems.

To this purpose, we intend to take into account a renown entanglement detection technique known as *witness* [9] [10]. Entanglement detection witnesses are special physical observables with the uncanny property that they are perfectly capable of recognizing a certain subset of entangled states and distinguish it from all separable states. The idea is simple: when we calculate the expectation value of one of these observables over a separable state, the outcome must satisfy some bounds, e.g. it takes only positive values. Therefore, when we obtain a negative expectation value we are ensured that the state we are considering is entangled.

In this work we are going to introduce an even more advanced concept: the *quantitative* entanglement witness (QEW). These operators are still physical observables capable of distinguishing between separable states and a certain class of entangled states, but they are able to do more than that. Indeed, when we get the expectation value of a QEW over a given state, we automatically obtain a *lower bound* on the degree of entanglement of that state; where this lower bound is a relatively simple function of the expectation value we extracted (of course, depending on the entanglement measure we are considering as well).

The reason why these witness-based techniques attract our attention is that they are real entanglement measurement methods, in the physical sense. So, according to this picture, our actual problem is to translate these special observables in terms of measurements which we know how to perform, concerning electron transport in solid nanostructures. Fortunately, there is a natural way of observing processes of electronic conduction in these systems, which is both a quite easy experimental task, and is particularly suitable for researching non-local properties, like entanglement. This tool is electrical current noise correlations measurement.

In definitive, the operative scheme we are actually going to present takes into account the capability of performing multiple measurements of current noise correlations, whose outcomes let us to fix a certain set of lower bounds on the entanglement of the unknown state, via quantitative entanglement witness

technique.

Clearly, our scheme has some physical requirements which must be satisfied by any experimental apparatus in order to be acceptable for its execution:

- we can control the conduction state by performing a certain set of *local* unitary transformations,
- we can split the flows of conducting carriers belonging to different channels [11] [12], and therefore measure single channel current intensities.

If these conditions are satisfied by a candidate physical system, then the scheme can be run.

In the first chapter we will explain the concept of witness as an operative entanglement detection technique, and exhibit a construction for some standard QEWs. Chapter 2 reviews fundamental properties concerning coherent electron transport in mesoscopic condensed matter structures; then we will discuss the idea of entanglement detection via noise correlation, and we will briefly present some already known proposals to approach this problem. In chapters 4 5 and 6 we explicitly explain our entanglement measuring scheme works, focusing our attention on which operations and current noise correlation measurements must be performed and why.

Chapter 1

Witnessing entanglement

1.1 Entanglement characterization

In this work we will focus our attention on some specific approaches to the problem of detecting and measuring entanglement on a bipartite physical system. Quantum entanglement has always been a very interesting property of quantum physics, yet only in recent decades people began to view it like a resource. The discovery of this new aspect launched intensive experimental efforts to produce quantum entangled states, and at the same time it stimulated the development of a rigorous and mathematically consistent analysis of this property.

What is essentially quantum entanglement? A standard textbook answer could be: "entanglement is a particular kind of quantum correlation whose properties and effects can not be emulated by means of any type of classical correlation or shared information". To explain what are these quantum correlations let us consider a bipartite physical system, i.e. a system composed by two distinguishable subsystems (indexed as A and B). Its Hilbert space takes the usual form of tensor product, $\mathcal{H}_d \otimes \mathcal{H}_d$; with $\mathcal{H}_{A,B}$ being the Hilbert space associated with A and B respectively. Separable or "non-entangled" states are defined as those density matrices of the form

$$\rho_{sep} = \sum_i p_i \left(|a_i\rangle \langle a_i| \otimes |b_i\rangle \langle b_i| \right), \tag{1.1}$$

with p_i being probabilities. It is easy to show that the separable states form a convex subset of the states space. They bear a relevant physical meaning as well: they are the most general class of states which can be prepared when the two quantum subsystems are not directly interacting. Entangled states are those states (either pure or mixed) which do not satisfy these property. Therefore, it is impossible to prepare entangled states only by means of *local operations and classical communication* (LOCC); entanglement is properly a quantum feature, thus quantum interaction is necessary in order to create it.

Analogously, when we operate a transformation on a bipartite state only by means of local (not necessarily invertible) operations and classical communication, we can never increase its entanglement: the entanglement of the final state shall be equal or lower than the starting one. According to this picture, it makes sense define a standard amount of entanglement which we will refer to as funda-

mental unit. Conventionally, this unit is often associated to the entanglement held by the *spin singlet state*

$$\left|\Psi_2^-\right\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{1.2}$$

We shall say that the spin singlet state carries 1 *bit of entanglement* (e-bit). This is, for example, the exact quantity of entanglement two distant users must share to teleport a qubit, via qubit teleportation protocol.

But how much any given state is entangled? Since we fixed a standard unit, we are now interested in a quantitative method of classifying entangled states. This can be done introducing *entanglement measures*.

Quantum physicists agreed on a particular set of fundamental requirements for entanglement measures in order to be acceptable. Among these conditions we recall: the decreasing monotonicity under LOCC operations, the continuity under the Hilbert norm, the additivity under tensor product, and the convexity under state mixing. Then, in order to satisfy these requirements, various different measures for entanglement were developed [1] [13] [14] [15] [16]. Here we present a pair of entanglement measures both for pure and mixed states in bipartite systems which suits particularly well our purposes.

Entanglement measures for pure states

- *Von Neumann entropy of the reduced density matrix*

First we write the density matrix $|\psi\rangle\langle\psi|$ of the pure state read in an orthonormal base of separable states. Then we compute the partial trace $\rho_A = Tr_B[|\psi\rangle\langle\psi|]$, i.e. we trace only on the degrees of freedom of one subsystem (no matter which one, the final result is the same). Finally, we compute the von Neumann entropy of ρ_A:

$$\mathcal{E}_{VN}(\psi) = -Tr(\rho_A \log \rho_A).$$

The idea of this entanglement measure is simple: since the more a pure bipartite state is entangled, the more its reduced density matrix is mixed, then evaluating the reduced density matrix purity gives us an information on the entanglement of the whole state. For example,the reduced density matrix of the spin singled state is the totally mixed state $\mathbb{I}/2$, whose Von Neumann entropy is exactly 1.

- *Negativity*

After writing the density matrix of the pure state read in an orthonormal separable basis, we perform the partial transpose, i.e. we transpose only on the degrees of freedom of one subsystem (again, no matter which one). Then we calculate:

$$\mathcal{N}(\psi) = \frac{\| \, |\psi\rangle\langle\psi|^{T_B} \| - 1}{d-1},$$

where the norm $\|\cdot\|$ is the commonly said 'trace norm', defined as follows: $\|O\| = Tr|O| = Tr\sqrt{O^\dagger O}$, or, equivalently, the sum of the moduli of the eigenvalues of O. It is easy to see that the partial transpose of a separable state is again a (separable) density matrix, and then its negativity is 0; while for the spin singlet we again obtain 1.

Entanglement measures for mixed states

- *Entanglement of formation*

This is a generalization for mixed states of the von Neumann entropy of the reduced density matrix. We know that any mixed state can be written as a convex combination of pure states: $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. The entanglement of formation is the averaged von Neumann entropy of the reduced density matrices of these pure states $|\psi_j\rangle$, minimized over all possible decompositions; essentially it is the convex roof extension of the von Neumann entanglement measure

$$\mathcal{E}_{OF}(\rho) = \inf_{\{dec\}} \sum_j p_j \;\; \mathcal{E}_{VN}(\psi_j).$$

It can be shown that the entanglement of formation of a given state ρ is the total amount of spin singlets which are needed to create ρ only by performing LOCC operations [1] [14] [15] [16].

- *Convex-roof extended negativity (CREN)*

This is the natural way to extend the negativity measure to mixed states, by means of the convex-roof. Thus, the CREN is the averaged negativity of the pure states that decompose ρ, minimized over all possible decompositions:

$$\mathcal{E}_{CREN}(\rho) = \inf_{\{dec\}} \sum_j p_j \;\; \mathcal{N}(\psi_j).$$

It is important to remind that all the measures described here require to have full knowledge of the state we are interested in. In the following sections we will focus on less powerful but simpler and operative tools for experimental detection and quantitative estimation of entanglement. Later we will make use of these techniques in the condensed matter models we are going to consider.

1.2 Entanglement witnesses

We are now going to introduce a tool for detecting entanglement known as *entanglement (detection) witness* [9] [10]. Witnesses are observables capable of distinguishing between a certain subset of entangled states and all possible separable states of the system. Given any entangled state ρ_E, an entanglement witness for that state is an hermitian operator $\mathbb{W}$, having the following property.

$$Tr[\rho_E \mathbb{W}] < 0 \qquad \text{and} \qquad Tr[\rho_{sep} \mathbb{W}] \geq 0, \quad \forall \rho_{sep} \;\; \text{separable}.$$

More in general, a witness is an observable whose expectation values on any separable state are bounded by some inequalities (in the previous example they are only positive valued), and there exist at least a state for which the outcome violates these bounds. Therefore, when we get an expectation value for the witness which is out of those bounds, then we may claim that the state we considered was entangled for sure.

Now, we already mentioned that entanglement is not an observable in the physical meaning; according to this scheme, we can translate this claim into the proposition which says that a *perfect witness* does not exist In other words, it is

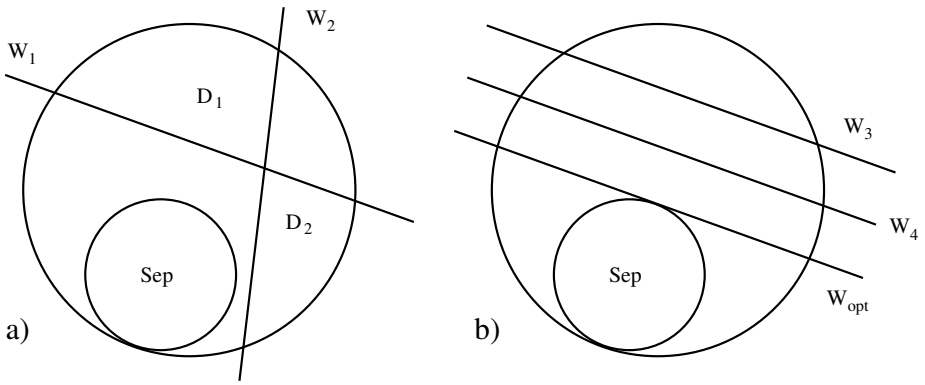

Figure 1.1: Schematic representation of entanglement detection witnesses onto the state space. a) W_1 (resp. W_2) detects entanglement for states belonging to D_1 (D_2), none of these two witnesses is strictly better than the other. b) W_{opt} is an optimal witness, and is better than W_4, which is anyway better than W_3.

impossible to define an observable whose expectation values take only positive values on separable states, only negative values on entangled ones. For instance, if we postulate the existence of an observable which is capable to see perfectly the difference between any separable and any entangled state, we immediately incur in an absurd.

To see this, suppose by contraction that we have a hermitian operator O on a bipartite system which, evaluated on any given state can tell us if that state is separable or entangled with perfect accuracy. Without loss of generality, we can describe this property as follows.

$$\begin{cases} \langle\psi|O|\psi\rangle \geq 0 & \text{if } \psi_s \text{ separable} \\ \\ \langle\psi|O|\psi\rangle < 0 & \text{if } \psi_e \text{ entangled.} \end{cases} \tag{1.3}$$

We now choose a pair of orthogonal vectors for each one of the two subsystems, that we will call $|a\rangle_j$ and $|b\rangle_j$. It follows directly by definition that $|aa\rangle$, $|ab\rangle$, $|ba\rangle$ and $|bb\rangle$ are separable state vectors, while $(|aa\rangle \pm |bb\rangle)$ are not. Then, by eq. (1.3) we must have

$$(\langle aa| \pm \langle bb|)\, O\, (|aa\rangle \pm |bb\rangle) < 0.$$

But this leads to the following absurd

$$\begin{aligned} \langle aa|O|aa\rangle + \langle bb|O|bb\rangle = (\langle aa| + \langle bb|)\, O\, (|aa\rangle + |bb\rangle) + \\ + (\langle aa| - \langle bb|)\, O\, (|aa\rangle - |bb\rangle) < 0. \end{aligned}$$

In other words, at least one between the expectation values of O calculated on the separable states $|aa\rangle$ and $|bb\rangle$ must be strictly negative. This is in contrast with the requirement of O taking only positive or null values on separable states.

There is also a curious, schematic way of thinking entanglement detection witnesses. We can imagine to draw the entire convex space of the bipartite mixed states, like in figure 1.1; the outside border of this region is composed

by pure states. If we take into account the subset of separable states, this is a convex set as well, and touches the border of the entire states space through pure separable states. Entanglement witnesses may also viewed according to this picture like straight cuts cuts in the states space, but they never divide the separable states subset. Its expectation values take positive outcomes on states standing in the same side which contains the separable ones; consequently we may detect entanglement in all the states belonging to the other side. We claim that a certain witness is 'better' than another one if it detects all the entangled states detectable by the other witness. This is clearly a partial ordering relation, and easily comprehensible within this image. We say that a witness is optimal if it does not exist a witness better than it. Optimal witnesses, in the picture, are cuts tangent to the separable states space.

1.3 Two special classes of states: isotropic and Werner states

In the present sections, we are going to present two classes of states, defined on an $\mathcal{H}_d \otimes \mathcal{H}_d$ state space, which show *symmetry* under certain groups of local transformations. Later we will show that these sets of states allow us to build some entanglement witnesses characterized by particular local-symmetry properties.

Isotropic states

Isotropic states are the first set we want to introduce. In order to define such states we choose an orthonormal basis for each of the two subsystems. We will refer to these basis states as $|\alpha\rangle_A$ and $|\alpha\rangle_B$ respectively, where the index α belongs to the discrete set $\{1..d\}$. In this picture, $|\alpha\beta\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$ is a basis of separable states for the entire bipartite system. The Isotropic states of $A+B$ are defined as *those states that are invariant under all transformation of the form* $U \otimes U^*$, where U is any unitary transformation acting on one subsystem, and the asterisk denotes complex conjugation of all the entries of U read in the chosen basis. That is

$$(U \otimes U^*)\rho_I(U \otimes U^*)^\dagger = \rho_I \tag{1.4}$$

The choice of the two subsystems basis provides a well defined mapping between operators in subsystem A and those in subsystem B, i.e. $M_A \to M_B$. Independently of d, the only pure state that satisfies the previous condition is the following:

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=1}^{d} |\alpha\alpha\rangle \ .$$

Moreover, it can be shown that all the mixed states that show such invariance are all those which can be written as $\rho_I = a\mathbb{I} + b|\Phi_d^+\rangle\langle\Phi_d^+|$, where $\mathbb{I}$ represents the fully mixed state (before normalization). Imposing positivity and normalization (i.e. $\langle\psi|\rho_I|\psi\rangle \geq 0$ for any pure state ψ, and $Tr[\rho_I] = 1$), we see that the set of Isotropic states is a one (real) parameter class of states

$$\boxed{\rho_I(g) = \frac{1}{d(d^2-1)}\left[(d-g)\mathbb{I} + (gd-1)\mathbb{G}_d\right], \qquad 0 \le g \le d,} \tag{1.5}$$

where

$$\mathbb{G}_d \equiv d\,|\Phi_d^+\rangle\langle\Phi_d^+| = \sum_{\alpha,\beta} |\alpha\alpha\rangle\langle\beta\beta|\,, \tag{1.6}$$

which manifests the property: $\mathbb{G}_d^2 = d\,\mathbb{G}_d$. They are a convex set, since by mixing two or more isotropic states we again obtain an element of this class. In this scheme g is a real parameter, and the positivity condition is satisfied as long as it ranges in the interval $[0, d]$. We see immediately that $\rho_I(1/d) = \mathbb{I}/d^2$ and $\rho_I(d) = \mathbb{G}_d/d$. A useful property of the set (1.5) of states is the following:

$$Tr[\,\rho_I(g)\,\mathbb{G}_d] = g \qquad or \qquad \langle\Phi_d^+|\rho_I(g)|\Phi_d^+\rangle = g/d.$$

In other words g has a straightforward physical meaning as well: it is the expectation value of the observable $\mathbb{G}_d$ on the Isotropic state $\rho_I(g)$ we are considering.

Another way to describe this set of states is the following

$$\rho_I(g) = \frac{d-g}{d(d^2-1)}\,\mathbb{P}_\Phi^\perp \;+\; \frac{g}{d}\,\mathbb{P}_\Phi,$$

where $\mathbb{P}_\Phi$ and $\mathbb{P}_\Phi^\perp$ are respectively the projectors onto $|\Phi_d^+\rangle$ and his orthogonal. In other words $\mathbb{P}_\Phi = \mathbb{G}_d/d$; $\mathbb{P}_\Phi^\perp = \mathbb{I} - \mathbb{P}_\Phi$.

Werner states

Werner states are another one real parameter class of special states of the system $A + B$. They are defined as *the set of states which are invariant under all transformation of the form* $U \otimes U$, where U is any unitary transformation acting on the single subsystem.

$$(U \otimes U)\rho_W(U \otimes U)^\dagger = \rho_W. \tag{1.7}$$

Like in the isotropic case, choosing an orthonormal basis for each subsystem gives us a natural mapping between operators acting on either A or B only; so that it makes sense performing on both of them the same unitary local transformation U.

In order to satisfy the previous equation the Werner states have to be of the form $\rho_W = a\mathbb{I} + b\mathbb{F}_d$, where

$$\mathbb{F}_d \equiv \sum_{\alpha,\beta} |\alpha\beta\rangle\langle\beta\alpha|, \tag{1.8}$$

is usually called "*swap* operator", as its role is to exchange the degrees of freedom between system A and B, preserving the state, i.e. $\mathbb{F}_d|\psi\rangle_{AB} = |\psi\rangle_{BA}$. We would like to remark that, differently from $\mathbb{G}_d$, $\mathbb{F}_d$ is not a positive operator: its spectrum is $\{-1, 1\}$, and of course $\mathbb{F}_d^2 = \mathbb{I}$. Let's keep in count positivity and normalization and we obtain:

$$\boxed{\rho_W(f) = \frac{1}{d(d^2-1)}\left[(d-f)\mathbb{I} + (fd-1)\mathbb{F}_d\right], \qquad -1 \leq f \leq 1.} \tag{1.9}$$

Now the interval in which our real parameter can take values is $[-1, 1]$. Note that for $d > 2$ there is no pure Werner state, for any f in the interval. Only in $\mathcal{H}_2 \otimes \mathcal{H}_2$ there exist a pure Werner state, it is clearly the singlet state of eq. (1.2); it occurs when $f = -1$.

Again the parameter f has a relevant physical meaning, being the expectation value of F_d on the selected state:

$$Tr[\,\rho_W(f)\,\mathbb{F}_d] = f\ .$$

Finally, we can recall another common way of describing this set of states:

$$\rho_W(f) = \frac{f+1}{d(d+1)}\left[\sum_{\alpha<\beta} |\Psi^+_{\alpha\beta}\rangle\langle\Psi^+_{\alpha\beta}| + \sum_{\alpha} |\alpha\alpha\rangle\langle\alpha\alpha|\right] + \frac{1-f}{d(d-1)}\sum_{\alpha<\beta} |\Psi^-_{\alpha\beta}\rangle\langle\Psi^-_{\alpha\beta}|$$

where we have written the Werner state as a convex combination of projectors onto states of the type $|\alpha\alpha\rangle$ and the 1-ebit entangled states

$$|\Psi^{\pm}_{\alpha\beta}\rangle = \frac{|\alpha\beta\rangle \pm |\beta\alpha\rangle}{\sqrt{2}}.$$

Entanglement properties of Isotropic and Werner states

Some of the great features that these classes of local-symmetric states satisfy involve the fact that computing certain measures of entanglement on Werner and Isotropic states is an *extremely easy* task. Recent works on this topic demonstrated that it is possible to write the entanglement, in terms of 'Entanglement of formation' and 'Convex-roof extended negativity', as a relatively simple function of d and g for Isotropic states, d and f for Werners. This result shows us at least two immediate advantages:

- We do not need to minimize over all possible pure decomposition to obtain the exact result, the calculus is straightforward
- As g and f are expectation values of observables, we can physically detect and measure entanglement.

Recently, B. Terhal and K.G. Vollbrecht [17] published an article where they explicitly calculate the $\mathcal{E}_{OF}$ for the Isotropic set of states, followed by a paper of Vollbrecht and R. Werner [18] where they compute the result for the Werner states. The CREN measure on those states has been calculated by S.Lee *et al.* in a work of 2003 [13]. We summarize briefly [19] their results.

- *Separability*
  - Isotropic states are separable for $0 \leq g \leq 1$, entangled for $1 < g \leq d$.
  - Werner states are separable for $0 \leq f \leq 1$, entangled for $-1 \leq f < 0$.

- *Entanglement of formation*
  - $\mathcal{E}_{OF}$ for the Werner state $\rho_W(f)$

$$\mathcal{E}_W(f) \equiv \mathcal{E}_{OF}\left(\rho_W(f)\right) = H_2\left(\frac{1+\sqrt{1-f^{\,2}}}{2}\right) \qquad for \;\; f \leq 0\,. \tag{1.10}$$

  - $\mathcal{E}_{OF}$ for the Isotropic state $\rho_I(g)$

$$\mathcal{E}_I(g) \equiv \mathcal{E}_{OF}\left(\rho_I(g)\right) = co[H_2(\gamma) + (1-\gamma)\log(d-1)] \qquad for \;\; g \geq 1\,. \tag{1.11}$$

  where

$$H_2(x) = -x\log x - (1-x)\log(1-x), \tag{1.12}$$

  and γ is a function of the Isotropic parameter,

$$\gamma(g) = \frac{1}{d^2}\left(\sqrt{g} + \sqrt{(d-1)(d-g)}\right). \tag{1.13}$$

  In eq. (1.11) '*co*' stands for *convex hull*, meaning the largest convex function that is nowhere larger than the given function. It is necessary to introduce this adjustment to force the function $\mathcal{E}_I$ to be convex, but only if $d \geq 3$; in the case $d = 2$ the expression is already convex.

- *Convex-roof extended negativity*
  - $\mathcal{E}_{CREN}$ for the Isotropic state $\rho_I(g)$

$$\overline{\mathcal{E}_I}(g) \equiv \mathcal{E}_{CREN}\left(\rho_I(g)\right) = \max\left\{0\,,\; \frac{g-1}{d-1}\right\} \tag{1.14}$$

  - $\mathcal{E}_{CREN}$ for the Werner state $\rho_W(f)$

$$\overline{\mathcal{E}_W}(f) \equiv \mathcal{E}_{CREN}\left(\rho_W(f)\right) = \max\left\{0\,,\; \frac{-f}{d-1}\right\} \tag{1.15}$$

1.4 Isotropic and Werner optimal witnesses

For both Isotropic and Werner classes of states, we already know operators that seem good candidates to be optimal witnesses, for any dimension d. We are speaking about the observables $\mathbb{F}_d$, for Werner states, and $\mathbb{I} - \mathbb{G}_d$ for Isotropics.

We already showed that these two operators behave jut like entanglement detection witnesses when we measure them on the class they belong to; this implies that if we demonstrate that these operators take positive expectation values over any separable state of the system, then they are real entanglement detection witnesses, onto the global state space.

To see this consider a generic separable state, defined by eq. (1.1), and compute:

$$Tr[\rho_{sep}\,\mathbb{F}_d] = \sum_{jkl} p_j\; \langle\psi_j|k\rangle\langle\phi_j|l\rangle\langle k|\phi_j\rangle\langle l|\psi_j\rangle = \sum_j p_j\,|\langle\psi_j|\phi_j\rangle|^2\;.$$

The result is obliviously positive, since it is a sum of positive terms. Similarly we have:

$$Tr[\rho_{sep}\,\mathbb{G}_d] = \sum_{jkl} p_j \;= \sum_j p_j \left|\langle\psi_j|\phi_j^\star\rangle\right|^2 \leq \sum_j p_j = 1\ ,$$

or equivalently, $\mathbb{I} - \mathbb{G}_d$ is positive on all separable states. It can be shown that these witnesses are optimal.

The above analysis shows that, if we manage to measure the expectation value on a state of $\mathbb{F}_d$ (resp. $\mathbb{G}_d$) and get a negative outcome (an outcome greater than 1), then we are sure that the state we are considering is entangled. Be aware that this condition is absolutely not necessary. Since there is no perfect witness, we know for sure the existence of entangled states which can not be detected from any of these witnesses.

Even if this is a promising result, we would like to do more than that: is it possible, for instance, not only recognize an entangled state, but also say something about how much entanglement it carries? Can Isotropic and Werner states give us a hint on that? We will be able to answer there question introducing the concept of *quantitative entanglement witness.*

1.5 Quantitative entanglement witness through twirl operation

Before presenting a definition of quantitative witness, we focus on the mathematical context that allows us to see why Isotropic and Werner states, just for the property of being invariant under a group of local symmetries, are so special to our purpose. According to these arguments we need to define an algebraic tool, known as the 'twirl' operation, that plays the role of an endomorphism on the mixed states space which is ruled by some local symmetry.

Now, consider Λ being a closed group of local unitary operators on $\mathcal{H}_d \otimes \mathcal{H}_d$, i.e. operators of the form $U = (U_1 \otimes U_2)$, and $U^\dagger U = UU^\dagger = \mathbb{I}$. As a closed subgroup of the unitary group, Λ is compact, hence it carries a unique measure that is normalized and invariant under left and right group translation. The meaning of this natural measure, known as '*Haar* measure' as well, is a sort of averaging over the elements of the group; it will be denote by $\int dU$.

The fundamental ingredient of our theory is the following transformation, that acts on the operators on $\mathcal{H}_d \otimes \mathcal{H}_d$:

$$\mathcal{P}_\Lambda(A) = \int_\Lambda dU (UAU^\dagger).$$

It is usually referred in literature as the 'twirl' operation, and it satisfies many remarkable properties. First of all, it acts like a projection, because $\mathcal{P}_\Lambda^2 = \mathcal{P}_\Lambda$ since the group Λ is closed. Then, it can be shown that this function takes positive operators to positive operators. Moreover, it is a *completely positive map*, i.e. any extension of this operation of the form $\mathcal{I}_k \otimes \mathcal{P}_\Lambda$ (where $\mathcal{I}_k$ is the map which takes any $k \times k$ matrix to itself), we again obtain a positive mapping [1]. Finally, it is also trace preserving; these three conditions let us say that $\mathcal{P}_\Lambda$ is a CPT map. In particular any density matrix goes to a density matrix via twirling. Using the invariance of the Haar measure it is immediately clear that

$\mathcal{P}_\Lambda(A) = A$ only if A belongs to the commutant of the group: $[U, A] = 0$ for any $U \in \Lambda$. Now, since $\mathcal{P}_\Lambda$ is a projector, it is clear that the commutant of Λ is also the range of the twirl operation. Moreover, we are guaranteed that the identity is always taken to itself.

On top of that we recognize that twirl is a LOCC transformations, i.e. that it can be physically achieved by means of local operation and classical communication. This simply follows from requiring as hypotheses that the entire group Λ were made by local unitary matrices.

Quantitative entanglement witness concept

The reason why we focused our attention on the previously described twirl operation and in particular to his LOCC property will be discussed here. When speaking about entanglement measures, there is a remarkable requirement that is absolutely necessary for any acceptable measure to be satisfied: we are talking about the *monotonicity under LOCC* operations. It is fundamental, for a good entanglement measure, that if I transform a given state by means of local quantum operations and classical communication, the entanglement measured on the final state must not be greater than in the beginning.

$$\mathcal{E}_?\left(\mathcal{T}_{LOCC}(\rho)\right) \leq \mathcal{E}_?(\rho)$$

must be satisfied for all transformations $\mathcal{T}_{LOCC}$ and states ρ, in order for any candidate $\mathcal{E}_?$ to be a good entanglement measure.

As far as we are concerned, it has been fairly demonstrated that the measures for entanglement we previously introduced (i.e. the 'entanglement of formation' and the 'convex roof entanglement negativity'), satisfy this requirement [13] [14] [15] [16]. Then we can conclude that

$$\mathcal{E}_{OF}\left(\mathcal{P}_\Lambda(\rho)\right) \leq \mathcal{E}_{OF}(\rho) \qquad \forall \text{ state } \rho$$

is true, simply asking that Λ were a group of *local and invertible* operators; the same statement is obviously valid for $\mathcal{E}_{CREN}$ too.

Consider then the following set: $\Lambda_{UU} = \{U \otimes U\}$. It can be easily shown that this is a group an it is closed, and thus there exists a twirl operation associated with it. As discussed before, $\mathcal{P}_{UU}$ contracts all the states space into the algebra of operators invariant under $U \otimes U$; but this algebra is the class of Werner states. Putting all together, we get a great result:

$$\mathcal{E}_{OF}(\rho) \geq \mathcal{E}_{OF}(\mathcal{P}_{UU}(\rho)) = \mathcal{E}_W(\overline{f}), \qquad \text{where } \overline{f} = Tr[\mathbb{F}_d\, \mathcal{P}_{UU}(\rho)].$$

We are now interested in studying the term $Tr[\mathbb{F}_d\, \mathcal{P}_{UU}(\rho)]$. By using the invariance of $\mathbb{F}_d$ under any operation of the group we may write $\mathbb{F}_d = (U \otimes U)\, \mathbb{F}_d\, (U \otimes U)^\dagger$. Then, recalling the definition of twirl, we see that

$$\begin{aligned} \mathbb{F}_d\, \mathcal{P}_{UU}(\rho) &= \int dU\ \mathbb{F}_d\, (U \otimes U)\, \rho\, (U \otimes U)^\dagger \\ &= \int dU\ (U \otimes U)\, \mathbb{F}_d\ \rho\, (U \otimes U)^\dagger = \mathcal{P}_{UU}(\mathbb{F}_d\, \rho)\ . \end{aligned}$$

Finally, we use the trace preservation property of the twirling:

$$\overline{f}\ =\ Tr[\mathbb{F}_d\, \mathcal{P}_{UU}(\rho)]\ =\ Tr[\mathcal{P}_{UU}(\mathbb{F}_d\, \rho)]\ =\ Tr[\mathbb{F}_d\, \rho].$$

In other words, the twirl $\mathcal{P}_{UU}$ takes any state in the unique Werner state which has the same expectation value of $\mathbb{F}_d$, but at the same time the degree of entanglement on the target state depends only on this expectation value. We are at last ready to approach the concept of quantitative witness, as it was firstly introduced by Vollbrecht and Werner [18]. A quantitative entanglement witness (briefly QEW) is an observable with the property that, when we read its expectation value on any state, gives us a lower bound on the degree of entanglement of that state, in terms of a relatively simple function of the outcome alone.

Theorem. *The observable $\mathbb{F}_d$ is a quantitative entanglement witness of the Werner type, i.e. for a given state ρ it holds:*

$$\boxed{\mathcal{E}_{OF}(\rho) \geq \mathcal{E}_W\left(Tr[\mathbb{F}_d\,\rho]\right).} \tag{1.16}$$

Now we may understand why this argument is so powerful: not only $\mathbb{F}_d$, being an observable, can be measured on a generic state and tell us, in the case of a negative result, that the state is entangled, but also that its entanglement (in terms of $\mathcal{E}_{OF}$) is *at least* a certain finite nonzero quantity. The same thesis holds also for CREN by replacing $\mathcal{E}_W$ with $\overline{\mathcal{E}_W}$ in the formula, and naturally on any other good entanglement measure, but the result is effective for those measures which show a simple expression when calculated on Werner states.

Note that no residual dependence on the twirl operation lasts in the final result, basically its mere existence is sufficient to lead us to this acknowledgement.

We are perfectly able to reproduce the previous computation, choosing, this time, the unitary local group of the form $U \otimes U^\star$, and see what happens. It is not hard to see that, making this choice, Isotropic states will be involved. The result is straightforward.

Theorem. *The observable $\mathbb{G}_d$ is a quantitative entanglement witness of the Isotropic type, i.e. for a given state ρ it holds:*

$$\boxed{\mathcal{E}_{OF}(\rho) \geq \mathcal{E}_I\left(Tr[\mathbb{G}_d\,\rho]\right).} \tag{1.17}$$

The demonstration is exactly the same presented before for Werner states. In fact it can be shown that $\Lambda_{UU^\star}$ it is still a group (as the function $U \to U^\star$ is an inner homomorphism). And then we obtain again that $\mathbb{G}_d\,\mathcal{P}_{UU^\star}(\rho) = \mathcal{P}_{UU^\star}(\mathbb{G}_d\,\rho)$, which leads to the same result as before.

So, let's resume what we have learned. We found two classes of locally symmetric states which are extremely simple to evaluate certain measures of entanglement on. Then we showed that for any generic, even unknown, state, we can establish two distinct lower bounds on its degree of entanglement simply by extracting its expectation values on a couple of physical observables. The following table synthesizes the results we got so far.

$$\boxed{\begin{array}{ll} \mathcal{E}_{OF}(\rho) \geq \mathcal{E}_W\left(Tr[\mathbb{F}_d\,\rho]\right) & \mathcal{E}_{CREN}(\rho) \geq \overline{\mathcal{E}_W}\left(Tr[\mathbb{F}_d\,\rho]\right) \\ \mathcal{E}_{OF}(\rho) \geq \mathcal{E}_I\left(Tr[\mathbb{G}_d\,\rho]\right) & \mathcal{E}_{CREN}(\rho) \geq \overline{\mathcal{E}_I}\left(Tr[\mathbb{G}_d\,\rho]\right). \end{array}} \tag{1.18}$$

1.6 Generating further witnesses

Although the conclusion we just found is surely remarkable, we may want to do better without involving further calculation. We built two different types of quantitative entanglement witnesses, which depend on the behaviour of an algebra of states invariant under a group of local symmetries (whether it is Werner-like or Isotropic-like), and we found two distinct observables, each one of them obeying the laws of one particular type of QEW. But now we recall, as discussed earlier in this chapter, that both Isotropic and Werner classes of states are dependent on a choice of a separable orthonormal basis. This suggest us the following idea: what if we choose a different basis? As a new set of Isotropic or Werner states would appear, should we find another QEW of the same type we described before? Well, the answer is yes, and now we are going to explain this principle in a correct mathematical formulation.

Let $\mathbb{Q}$ be a Werner type quantitative entanglement witness, so that $\mathcal{E}_{OF}(\rho) \geq \mathcal{E}_W\left(Tr[\mathbb{Q}\,\rho]\right)$, and let ρ be a generic given density matrix. Consider the following local unitary transformation $U_1 \otimes U_2$; the $\mathcal{E}_{OF}$ measure calculated on a state before and after applying the transformation must take the same value, i.e. it holds:

$$\mathcal{E}_{OF}(\rho) = \mathcal{E}_{OF}\left((U_1 \otimes U_2)\,\rho\,(U_1 \otimes U_2)^\dagger\right) \ ,$$

because it is a LOCC operation and it is invertible.

We obtained a new state, $\widetilde{\rho} = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^\dagger$ on which we apply the Werner QEW property of $\mathbb{Q}$.

$$\mathcal{E}_{OF}(\widetilde{\rho}) \geq \mathcal{E}_W\left(Tr[\mathbb{Q}\,\widetilde{\rho}]\right) = \mathcal{E}_W\left(Tr\left[\mathbb{Q}\,(U_1 \otimes U_2)\,\rho\,(U_1 \otimes U_2)^\dagger\right]\right)$$

$$= \mathcal{E}_W\left(Tr\left[(U_1 \otimes U_2)^\dagger\,\mathbb{Q}\,(U_1 \otimes U_2)\,\rho\right]\right)$$

Where we used the ciclicity of the trace under multiplication order of matrices, or equivalently $Tr[AB] = Tr[BA]$. Then if we call $\mathbb{Q}' = (U_1 \otimes U_2)^\dagger\,\mathbb{Q}\,(U_1 \otimes U_2)$ we obtain, putting all together:

$$\mathcal{E}_{OF}(\rho) \geq \mathcal{E}_W\left(Tr\left[(U_1 \otimes U_2)^\dagger\,\mathbb{Q}\,(U_1 \otimes U_2)\,\rho\right]\right) = \mathcal{E}_W\left(Tr[\mathbb{Q}'\,\rho]\right) \tag{1.19}$$

We discovered another QEW of the same type, and it is $\mathbb{F}'$. Note that we only used the property of $U_1 \otimes U_2$ being invertible and local, which means that this demonstration does not depend on the type our witness belongs to; and as a consequence of that it works perfectly even for Isotropic QEWs.

Theorem. *If $\mathbb{Q}$ is a Werner type (resp. Isotropic type) quantitative entanglement witness, then for any couple of unitary transformation U_1 and U_2 acting on single subsystems, the observable $(U_1 \otimes U_2)\,\mathbb{Q}\,(U_1 \otimes U_2)^\dagger$ is still a QEW of the Werner (Isotropic) type.*

We acknowledged a very useful extension of the previous result with practically no further efforts. This ensures that all observables of the form $(U_1 \otimes U_2)\,\mathbb{G}_d\,(U_1 \otimes U_2)^\dagger$ are Isotropic QEWs and those which can be written like $(U_1 \otimes U_2)\,\mathbb{F}_d\,(U_1 \otimes U_2)^\dagger$ are Werner ones. Not only, but the physical meaning is remarkable as well. Assume we are measuring entanglement via quantitative witnesses, and manage to obtain the expectation value needed to compute a

single lower bound. Now, provided that our experimental device is able to reproduce always the same prepared state, we may repeat the entire experiment with the same identical setup, the only difference is that we apply a unitary local transformation before making the measurements. What we obtain by doing so is additional information about the entanglement of the state, in form of new, distinct and independent lower bounds on its degree of entanglement. Any time we run the experimental setup we may extract new information, the only requirement is tunable local control (by means of unitary transformations) of the state.

This is one of the reasons why entanglement witnesses are so appreciated candidates for good estimation of entanglement measurement. This technique for detecting and estimating entanglement is the tool which will allow us to investigate upon quantum correlations in condensed matter systems.

Chapter 2

Coherent transport theory

2.1 Condensed matter and quantum information

In the last decades quantum information focused its attention on quantum optical systems as a physical experimental context to apply its theories. Using polarization as internal degree of freedom, photons were an excellent experimental prototype of the qubit, easy to prepare in large quantities, and coherent for long distances and time intervals.

Only in recent times, quantum information began to consider also *condensed matter physical systems* as another potential experimental setting to apply its theories; in particular electron transport phenomena in metals or semiconductors. Indeed, physical research has developed a deep theoretical knowledge about mesoscopic solid state systems [20] [21], including coherent transport. This achievements lead to a mature solid state technology, characterized by building techniques for nanostructured solid devices which offer efficiency and, above all, *scalability*. If in future it will be possible to build quantum computers, they shall be probably based on solid state systems rather than optical systems; this because it will be more efficient to assemble a quantum calculator if its components are scalable. This is one of the reasons that suggested many physicists, to choose condensed matter mesoscopic system as candidates for certain experimental applications of quantum information studies [22].

In this chapter we review some basic properties belonging to the standard formulation of coherent electron transport in mesoscopic solid state structures, in order to give the reader a general view of the matter.

2.2 Landauer formulation of coherent transport

The study of electric conduction in mesoscopic solid state bodies gathered a lot of interest in the last decades. The approach which gathered more credit is the model developed by R. Landauer, M. Buttiker *et al* [23] [24], who proposed a picture for the electronic transport, in terms of transmission through coherent centers of scattering. This formulation, since it describes a very large variety of different solid systems (both metals and semiconductors), appeared to be

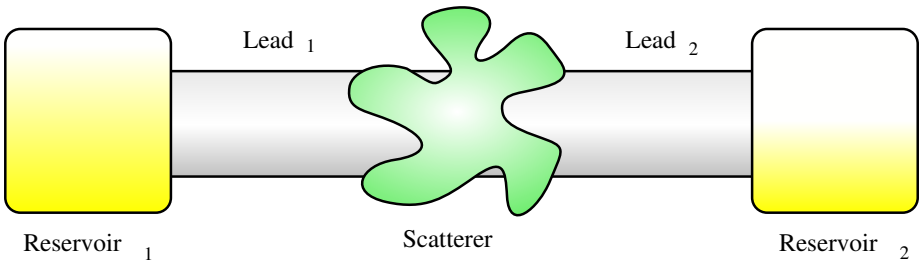

Figure 2.1: Electron conductance by transmission through a coherent center of scattering.

suitable in many physical contexts [20] [21]. As a reminder for the reader, we review the basic concepts that support this formulation.

The model

We suppose to have two electron reservoirs, separated from a solid state sample, whose physical dimensions are smaller than electron coherence length in that medium (i.e. we assume that conduction electrons undergo coherent motion). Each of these two reservoir is locally at the thermodynamical equilibrium, showing two temperatures T_L and T_R and two different chemical potentials μ_L and μ_R. In our scheme we may state that these reservoirs are so big that their thermodynamical properties may not be influenced by transport processes: in other words reaching their reciprocal equilibrium is forbidden, and their temperatures and Fermi levels will not change over time.

According to this theory, the sample is viewed like a *center of elastic scattering* which is attached to the two reservoirs by two *leads*, i.e. two zones where electrons undergo a free evolution, like in figure 2.1. Let us discuss the dynamic and the statistic of electrons in these leads for first.

The system of all carriers inside a single lead is formulated as a *non interacting electron gas*. The masses of these electrons are usually renormalized due to crystal lattice effects. This gas is confined in the transverse direction (perpendicular to the conduction direction) by a potential profile, e.g. a hard wall confinement. In the longitudinal direction the motion of all electrons is free, since the assume the confining potential being invariant under longitudinal translations. Therefore, transverse and longitudinal dynamic can be treated independently. For the first we have a discrete set of bound states, which are called transverse modes or subbands, while for the latter we have a continuous set of plane waves. The total energy of an electron level is therefore

$$E_n(k_\parallel) = \frac{\hbar^2}{2m_e^\star} k_\parallel^2 \; + \; \epsilon_n^\perp \; , \tag{2.1}$$

with $\epsilon_n^\perp$ being the energy of the n-th transverse mode, $k_\parallel$ the longitudinal wavevector, and $m_e^\star$ the electron effective mass. Thus, inside a lead, single electron levels are uniquely defined if we fix their subband n, longitudinal wavevector $k_\parallel$ and spin s, or equivalently if we set n, s and their total energy E and direction of motion $\leftrightarrow$ (which is the sign of $k_\parallel$).

We assume that the occupation the electrons levels in the leads whose direction is towards the scatterer is exactly the Fermi statistic, with T and μ of the reservoir they are connected to. Instead, the filling of *outcoming* electron levels is defined by applying the scattering process, by means of a transfer matrix, to the incoming state.

Statistical outcomes for operators

In the setting described above we are typically interested in statistical expectation of observables, $O_{\leftrightarrow}(E)$, which depend upon the energy and possibly the direction. We have to calculate it inside a single lead, for example the left one. Let us write separately the contribution given by all the incoming electrons $\langle O_{\rightarrow}\rangle_L$ and that given by all the outcoming ones $\langle O_{\leftarrow}\rangle_L$.

$$\langle O_{\rightarrow}\rangle_L = \sum_n \int_{-\infty}^{+\infty} dE \;\; f_{FD}(E,\mu_L,T_L) \;\; \nu_n(E) \;\; O_{\rightarrow}(E)$$

$$\langle O_{\leftarrow}\rangle_L = \sum_n \int_{-\infty}^{+\infty} dE \;\; \{(1-\tau_n(E)) \;\; f_{FD}(E,\mu_L,T_L) + \\ + \tau_n(E) \; f_{FD}(E,\mu_R,T_R)\} \;\; \nu_n(E) \;\; O_{\rightarrow}(E)$$

Basically we are calculating the product between the observable spectral value and the normal mode occupation, then we sum over all the normal modes. In this expression $\tau_n(E)$ (resp. $1-\tau_n(E)$) is the transmission (reflection) probability at the scattering center for the electron orbital of energy E, subband n. f_{FD} is the Fermi Dirac distribution, describing the lead statistic,

$$f_{FD}(E,\mu,T) = \frac{1}{1+\exp(\frac{E-\mu}{k_B T})} \; ; \tag{2.2}$$

and $\nu_n(E)$ is the *density of states*, describing the lead dynamic, which obliviously keeps a dependence on the transverse mode index:

$$\nu_n(E) \;\; = \;\; 2\,\frac{1}{2\pi}\,\frac{dk_{\|}}{dE} \;\; = \;\; \frac{L\sqrt{2m_e^{\star}}}{h}\,\frac{1}{\sqrt{E-\epsilon_n^{\perp}}} \; , \tag{2.3}$$

where we have multiplied by 2 due to the spin degeneracy. To obtain the final result of $\langle O\rangle$, its two components should be summed or subtracted, either $O_{\leftrightarrow}$ is symmetric or antisymmetric under change of direction (usually either is a scalar or vector variable). Just consider the case with $O_{\leftrightarrow}$ antisymmetric, or vector. Then if we write $\langle \underline{O} \rangle = \langle O_{\rightarrow} \rangle - \langle O_{\leftarrow} \rangle$, we find that

$$\langle \underline{O} \rangle_L = \langle \underline{O} \rangle_R = \sum_n \int_{-\infty}^{+\infty} dE \;\; \tau_n(E) \times \\ \times \{f_{FD}(E,\mu_L,T_L) \;\; - \;\; f_{FD}(E,\mu_R,T_R)\} \;\; \nu_n(E) \;\; O_{\rightarrow}(E)$$

Coherent charge current

Let us apply the previous calculation to the operator *density of electric current*,

$$\underline{j}(E) = \frac{e}{S}\,\underline{v} \;=\; \frac{e}{\hbar\, S}\,\frac{dE}{dk}\,.$$

with S the transverse section surface of the sample. We find that the velocity $\underline{v}$ cancels out with the density of states, it remains only a step function dependent on the transverse mode energy.

$$\langle j\rangle = \frac{2e}{h\,S}\sum_n \int_{-\infty}^{+\infty} dE\;\; \tau_n(E)\,\{f_{FD,L} - f_{FD,R}\}\,\vartheta(E-\epsilon_n^{\perp})\,.$$

Furthermore, if $\tau_n(E)$ is independent of n, we may sum the functions θ over n. We obtain the function $M(E) \equiv \sum_n \vartheta(E-\epsilon_n^{\perp})$ which counts the number of disposable transverse modes at a determined energy. By multiplying the equation by S we obtain the total *intensity of electric current* which flows through the lead.

$$\boxed{\langle I\rangle = \frac{2e}{h}\int_{-\infty}^{+\infty} dE\;\left[f_{FD}(E,\mu_L,T_L) - f_{FD}(E,\mu_R,T_R)\right]\;\tau(E)\;M(E)\,.} \qquad (2.4)$$

This is the final result, usually referred as *Landauer formula*. The strength of this expression is that all the information on the conduction properties of the sample is condensed into the elements of a scattering matrix, which appear in the expression in terms of spectral transmission coefficients $\tau_n(E)$. And so any *normal medium* (metallic conductors, semiconductors, insulators) as well may all be described according to this formulation.

Let us consider both reservoirs being at zero temperature. Clearly the Fermi Dirac distribution takes the form of the characteristic step function, and thus we have:

$$\langle I\rangle = \frac{2e}{h}\int_{\mu_1}^{\mu_2} M(E)\,\tau(E)\,dE.$$

Obviously $\mu_2 - \mu_1 = eV$ where V is the electric potential bias we keep between the contacts. As a further restriction, we assume that $M(\mu_1) = M(\mu_2)$, i.e. the energy range is small enough that the number of transverse modes is fixed over the whole interval. And we assume also the transmission through the scattering centre being energy independent. The well-known result we obtain is the following:

$$\boxed{I = \frac{2e^2V}{\hbar}M\,\tau \qquad \text{or also} \qquad R = \frac{1}{G} = \frac{h}{2e^2 M\,\tau},} \qquad (2.5)$$

where I is the average intensity, R the resistance, and G is the conductance of the sample. The quantity, $2e^2/h$ is called *quantum of conductance*: it is a very important result that it takes a finite value even when the conduction is ballistic ($\tau = 1$). Indeed, the nonzero resistance, and the associated voltage drop, takes place at the contacts, because we must do some work in order to maintain the chemical disequilibrium between the reservoirs. It is the operation of closing the electric circuit itself that cause a dissipation of energy.

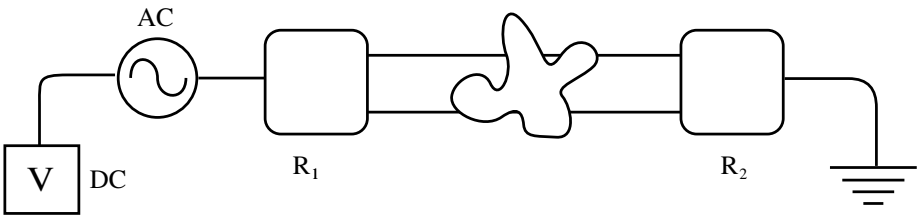

Figure 2.2: Right contact is put at ground potential, while electrical potential in the left contact is given by a contribution of a direct and an alternate components.

Differential conductance through lock-in technique

In order to obtain the equation (2.5), we made the assumption to be working with a sufficiently small voltage bias V. It is possible to overcome this approximation, by considering *differential conductance* in lock-in setups [25].

Let us assume, for instance, that the electrical potential at the left contact is the sum of a fixed constant V and a small harmonic oscillation $v\cos(\omega t)$ at low frequency ω; at the right contact the potential is zero. Therefore, the current intensity shall be a function of time.

$$\langle I\rangle(t) = \frac{2e}{h}\int_{-\infty}^{+\infty} dE\ \frac{1}{1+\exp(\frac{E-eV+ev\cos(\omega t)}{k_BT})}\ \tau(E)\ M(E)\ .$$

We now consider the time derivative of this expression, and we obtain

$$\frac{d\langle I\rangle}{dt} = -\frac{2e^2v}{h}\omega\sin(\omega t)\int_{-\infty}^{+\infty} dE\ \ \frac{d}{dE}\left[f_{FD}(E,eV,T)\right]\ \tau(E)\ M(E)$$

If we now set zero temperature, since f_{FD} is a step function, its energy derivative is a delta function

$$\frac{d\langle I\rangle}{dt} = \frac{2e^2v}{h}\omega\sin(\omega t)\int_{-\infty}^{+\infty} dE\ \delta E - eV\ \tau(E)\ M(E)\ = \\ = \frac{2e^2v}{h}\tau(eV)M(eV)\left[\omega\sin(\omega t)\right] \quad (2.6)$$

And we obtain a result which is basically equivalent to (2.5). In figure 2.2 we draw a schematic representation of the lock-in technique execution apparatus.

2.3 Multiterminal device: second quantization approach

We now include the possibility of having a *multiterminal* nanostructured device, allowing electron exchange with multiple external contacts. According to Buttiker model [20] [21] [23] [24], we think the sample as a common center of scattering; connected to the reservoirs through multiple leads, one per contact.

Therefore to identify a lead electron level in this system, we also have to specify which lead we are referring to.

The relevant degrees of freedom of this setting are summarized in the following list:

- Port j
- Energy E
- Direction $\leftrightarrow$
- Subband n
- Spin s

Our study will be focused on states which are entangled for spin and transverse mode degrees of freedom. We will refer to them together with the name of *channels*, regardless of their original type. So, it will be exactly on the channel degrees of freedom that we are going to perform our quantum informational approach. In fact, this is not the more general case, even states entangled for the direction and the energy degrees of freedom have been taken into account in literature.

Before moving further, we would like to include a brief remark. By eq. (2.1) it is clear that the number of open transverse modes at a given energy depends on the energy level itself, since the n-th subband is disposable if $\epsilon_n^\perp \leq E$. This simply tells us that in general the dimension of the transverse mode space, and therefore the number of channels as well, may be dependent on the energy level we are considering. Anyway, we usually make the assumption that the active range of energies, i.e. those involved in the transport process, is small enough to let us consider the number of disposable channels fixed over the whole energy interval.

Since, from now on, we are going to study entangled electron of conduction states, we need a language which allows us to describe microscopically a many-body non interacting fermion system. The second quantization formalism suits perfectly our needings [22]. We shall use it to define the many-electrons state inside the leads, for which we already found the one electron levels, see eq. (2.1).

To this purpose, let $\hat{a}_{j,\alpha,\leftrightarrow}(E)$ (resp. $\hat{a}^\dagger_{j,\alpha,\leftrightarrow}(E)$) be the Fermi operator which destructs (constructs) a carrier inside the j-th lead, in the level identified by channel α, energy E, and direction $\leftrightarrow$. From the statistical properties of fermions, these operators must undergo the usual anticommutation relations. For a continuous energy spectrum we have

$$\left\{\hat{a}_{j,\alpha,\rightarrow}(E)\ ,\ \hat{a}^\dagger_{k,\beta,\rightarrow}(E')\right\} = \left\{\hat{a}_{j,\alpha,\leftarrow}(E)\ ,\ \hat{a}^\dagger_{k,\beta,\leftarrow}(E')\right\} = \\ = \delta_{j,k}\ \delta_{\alpha,\beta}\ \frac{\delta(E-E')}{\nu(E)}\ , \quad (2.7)$$

where $\{O,P\} \equiv OP + PO$. The anticommutators for all the other combination of Fermi operators give a null outcome; $\nu(E)$ is the spectral density of states in the lead. We can consider a discrete energy spectrum as well, in such case case ν is not needed, and we will obtain simply $\delta_{j,k}\ \delta_{\alpha,\beta}\ \delta_{E,E'}$.

2.4 Intensity of current operator

Our aim is to use current correlators in order to detect channel entanglement. To this purpose, we now review the calculation of the electric current intensity operator for a single lead, written in the language of second quantization. We also manage to keep track of time dependence.

For the moment, we consider the contribution of electrons belonging to a given subband only, say α, and we use particle operators indexed by the longitudinal wavevector k, instead of E and the direction. For instance, $\hat{a}_{k,\alpha}$ (resp. $\hat{a}^\dagger_{k,\alpha}$) destructs (constructs) a carrier in the plane-wave orbital with wavevector k, and channel α. Now, let us consider the associated field operators $\hat{\xi}_\alpha(x)$ at position x, defined as follows

$$\hat{\xi}_\alpha(x) = \frac{1}{2\pi\sqrt{S}} \int \hat{a}_{k,\alpha}\, e^{ikx}\, dk. \tag{2.8}$$

We simply made a Fourier transform, adding S, the section surface of the lead, as a further normalization. Now we start from the (non-relativistic) electric current density operator in the field language, and we keep separate the contributions of the various channels.

$$\hat{j}_\alpha(x) = \frac{e\hbar}{2i\, m_e^\star} \left[\hat{\xi}^\dagger_\alpha(x) \left(\nabla\hat{\xi}_\alpha(x)\right) - \left(\nabla\hat{\xi}^\dagger_\alpha(x)\right) \hat{\xi}_\alpha(x)\right] \tag{2.9}$$

By substituting the field operators with eq. (2.8), we obtain

$$\hat{j}_\alpha(x) = \frac{1}{2\pi} \int \widetilde{j}_{q,\alpha}\, e^{iqx}\, dq\,,$$

where

$$\widetilde{j}_{q,\alpha} = \frac{e\,\hbar}{2\pi m_e^\star\, S} \int \hat{a}^\dagger_{k,\alpha}\, \hat{a}_{k+q,\alpha}\, k\, dk. \tag{2.10}$$

If we multiply by S we obtain again the total current intensity I. Let us translate this result in terms of energy and direction, instead of k. We can keep the energy spectrum continuous or discrete, which one of the two solutions suits more our needings. Here is the natural change of variables.

$$\frac{e\hbar}{2\pi m_e^\star} \int k\, dk \ldots \quad\longrightarrow\quad \frac{e}{h} \int dE \ldots \quad\longrightarrow\quad \frac{e}{h} \sum_E \frac{1}{\nu(E)} \ldots$$

If the energy range we are using is small enough we may also think the density of states ν as a constant rather than a function, usually we will adopt this approximation. In conclusion we obtain a quite simple expression for the *single channel spectral current intensity.*

$$\boxed{\widetilde{I}_\alpha(\omega) = \frac{e}{h\nu} \sum_E \left\{\hat{a}^\dagger_{\alpha,\rightarrow}(E)\, \hat{a}_{\alpha,\rightarrow}(E+\hbar\omega) \;-\; \hat{a}^\dagger_{\alpha,\leftarrow}(E)\, \hat{a}_{\alpha,\leftarrow}(E+\hbar\omega)\right\}.} \tag{2.11}$$

It is clearly visible the separate contribution of the particles in motion to the right and those in motion to the left (respectively $\hat{a}_\rightarrow$ and $\hat{a}_\leftarrow$). If we are interested in finding the time depending channel current operator instead, we obtain

$$\hat{I}_\alpha(t) = \sum_\omega e^{i\omega t}\, \widetilde{I}_\alpha(\omega)\ . \tag{2.12}$$

It is straightforward that if we want the *total* current (either spectral or time dependent) flowing through the lead, we should simply sum the previous results, (2.11) (2.12), over all the channels.

As a concluding remark, let us show that when we average $I(t)$ over long time intervals, we simply obtain $\widetilde{I}(\omega)$ for $\omega = 0$, i.e. the operator which counts the occupation number of all orbitals (with sign according to the direction).

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \hat{I}(t)\, dt \;=\; \sum_\omega \widetilde{I}(\omega)\, \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i\omega t}\, dt \;=$$
$$= \sum_\omega \widetilde{I}(\omega)\, \lim_{T\to\infty}\frac{\sin(\omega T)}{\omega T} \;=\; \sum_\omega \widetilde{I}(\omega)\, \delta_{\omega,0} = \widetilde{I}(0)\ .$$

It is straightforward to see that if we build a lead setup like in figure 2.1 and set the contacts to be thermal, we obtain exactly the Landauer result for $\widetilde{I}(0)$.

2.5 Noise correlators

The basic principle of our proposal for detection of entanglement in condensed matter systems, is finding correlations in the fluctuations of currents flowing through different leads. According to this picture, the *double frequency current noise correlator* between leads j and k is usually considered in literature [5] [6] [24] [32] [33]. It is defined as follows

$$\mathcal{S}_{j,k}(\omega,\omega') = \lim_{\mathcal{T}\to\infty} \frac{h\,\nu}{\mathcal{T}^2} \iint_0^{\mathcal{T}} e^{-i(\omega t+\omega' t')}\, \langle \delta\hat{I}_j(t)\, \delta\hat{I}_k(t')\rangle\, dt\, dt'\ , \tag{2.13}$$

where $\delta I_j = I_j - \langle I_j\rangle$ is the current fluctuation, and $\langle\,\cdot\,\rangle$ means taking the expectation value over the whole lead many-electron state. It is easy to see that in terms of spectral currents this correlators becomes simply

$$\mathcal{S}_{j,k}(\omega,\omega') = h\,\nu\, \langle \delta\widetilde{I}_j(\omega)\, \delta\widetilde{I}_k(\omega')\rangle \tag{2.14}$$

In this thesis, we will more often use the notation on the right side of this equation to identify current noise correlators. Therefore, to execute any of the detection schemes based on this principle (we will present some of them in the next chapters) an experimental apparatus must be capable of measuring:

- Single port currents, either time dependent $\langle\hat{I}(t)\rangle$ or spectral $\langle\widetilde{I}(\omega)\rangle$.
- Double port noise current correlators $\langle\delta\widetilde{I}_j(\omega)\, \delta\widetilde{I}_k(\omega')\rangle$

The same argument is equally valid if we are interested in single channel currents and noise correlators, simply we select a pair of channels α and β (one for each lead), and use the channel current intensity operators $\hat{I}_{j_\alpha}$ and $\hat{I}_{k_\beta}$ instead of the total intensities.

Often, in this work, we will speak about *shot-noise* correlator $\langle\delta\widetilde{I}_j\ \delta\widetilde{I}_k\rangle_{\underline{00}}$, by this we mean only the $\omega = \omega' = 0$ component of the noise current correlator.

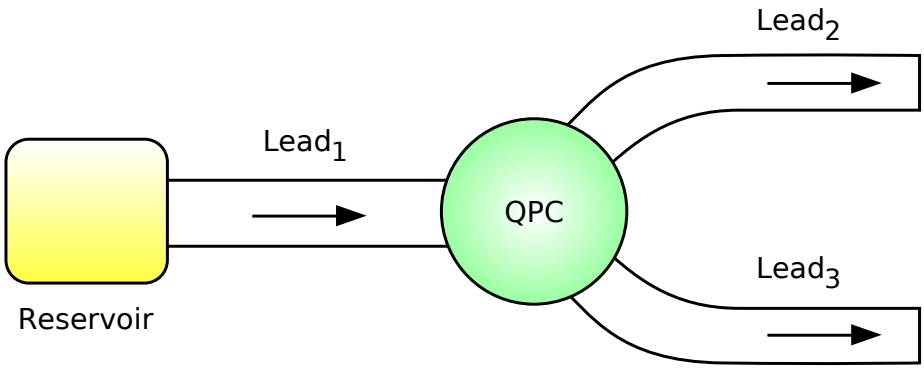

Figure 2.3: Schematic diagram of the device setup for Beenakker channel entanglement preparation protocol. Lead 1 is connected to a reservoir at $T = 0$, a quantum point contact splits the electrons incoming from the left into two possible outcoming leads. Backscattering is not allowed.

2.6 How to generate channel entanglement

A reasonable question that may be suggested at this point is if we are really able to prepare conduction states which are entangled for some channel degree of freedom. To this purpose, we now present as an example an experimental protocol proposed by C. Beenakker *et al* [2] [3] [4], which aim is EPR pair production in a nanostructured coherent conductor. Let us explain briefly how this device is supposed to works.

We have a 2D mesoscopic solid system under quantum Hall regime, built as in figure 2.3. Two edge states are open for coherent transmission; they represent all the channel degrees of freedom we intend to consider. Since they represent a 2 dimensional system, we will often refer to them as "pseudospin". Lead 1 is connected to an external contact, at zero temperature, which injects carriers in the system. A quantum point contact (QPC) splits the electron of conduction flow into two outcoming leads, labeled 2 and 3, backscattering is forbidden.

Let us write the conduction state inside the first lead, which is moving towards the scatterer (to the right).

$$|\Psi_{in}\rangle = \prod_{\mu<E<\mu+eV} \hat{a}^{\dagger}_{1,\uparrow}(E)\, \hat{a}^{\dagger}_{1,\downarrow}(E)\, |0\rangle \tag{2.15}$$

Now we apply the transformation given by the scattering matrix S, which represents the QPC, in order to obtain the outcoming state. Since the scattering process must be elastic the S matrix must be block-diagonalized in energy, i.e. it can not mix levels with different energies. If we now assume that the transmission probabilities 1-2 and 1-3 do is not influenced by the pseudospin, then the outcoming state shall be of the form

$$|\Psi_{out}\rangle = \prod_{E} \Big\{ (1-\tau)\, |\uparrow\downarrow\rangle_2 \; + \; e^{2i\varphi}\tau\, |\uparrow\downarrow\rangle_3 \; + \; -e^{i\varphi}\sqrt{2\tau(1-\tau)}\; \frac{|\uparrow\rangle_2 |\downarrow\rangle_3 - |\downarrow\rangle_2 |\uparrow\rangle_3}{\sqrt{2}} \Big\}, \tag{2.16}$$

where $|\uparrow\rangle_j$, $|\downarrow\rangle_j$ are outcoming electron states in the lead j, with pseudospin respectively $\uparrow$ or $\downarrow$, while $|\uparrow\downarrow\rangle_j$ means a pseudospin singlet entirely localized in lead j. Moreover, τ (resp. $1-\tau$) is the transmission probability of single electron to lead 3 (lead 2), while φ is a meaningless phase shift, in principle they could be a both function of the energy level E.

Now we may see the presence of three distinct contributions in eq. (2.16). We have the double transmission into lead 2, and double transmission into lead 3. The third terms describes the situation of a pair of electrons, one per lead, whose overall pseudospin state is a singlet, as in eq. (1.2).

Due to superselection rules related to particle conservation, within a given state all the components which share the same numbers of particles, also known as *sectors*, contribute separately to the global distillable entanglement of the state [4] [26] [27] [28], according to post-selection arguments (otherwise we incur into fluffy-bunny entanglement effects). Therefore, to calculate the total entanglement of the outcoming state we should compute the weighted sum among the entanglement quantities of the three distinct terms in eq. (2.16). The first two terms give zero contribution, since the two electrons are forced into the pseudospin singlet by Fermi statistical rules. The third term brings the only nonzero contribution, and its exact value is 1, because it is a Bell state. Therefore we have

$$\mathcal{E}_{OF}(\Psi_{out}) = 2\tau(1-\tau), \tag{2.17}$$

which reaches its maximum value when the electrons have the same probabilities to transmit into either lead 2 or lead 3, i.e. when $\tau = 1/2$.

In conclusion, we presented a protocol for the preparation of channel entangled states in nanostructured conducting solid systems. We would like to remark that we started having a thermal state, incoming toward the scattering centre, and what we obtained is that the outcoming state is non-thermal, and carries entanglement.

The protocol we reviewed is just a candidate for channel-entanglement preparation; there are other recent works which approach the same problem, for example refs. [7] [29] [30].

Chapter 3

Entanglement detection strategies I

In this chapter we review some theoretical schemes for detection of channel entanglement in coherent electron transporting nanostructures, which were already proposed in recent papers. To begin with, we present the most general shape for a conduction state, potentially channel-entangled, we intend to take into account in our study.

3.1 The black box

Let us assume we have a device able to inject a conduction state into a pair of coherent leads, spatially separated. The many-body electron state this device is able to prepare is always the same, the final purpose of our thesis is to investigate if and how much this prepared conduction state is entangled. We have very little information about the functioning of this device and can not interfere with it, for these reasons we shall call it the *black box*, or *entangler*. We are allowed to operate on the outcoming state, apply eventual transformations, and then measure noise current correlations. Basically we want to discover how much our black box is adequate to produce quantum entanglement on channel degrees of freedom.

To this purpose, we try to give a physical description of the conduction state outcoming from the entangler. If we want to be capable of manipulating and working efficiently with this state, we will be forced to make some assumptions on its form. Precisely, we will set the following requirements to be satisfied.

- Inside each port there is exactly only one carrier per every energy level open to transport.
- The state restricted to a single energy level is exactly the same for all energies open to transport.

The requirements just listed, do carry a physically relevance. The first one ensure us that we will incur neither in superselection rules nor in fermion statistical issues. The second condition tells us that we are going to apply our quantum information approach only on a single $d \times d$ dimensional state. The

global conduction state simply is a set of identical copies of this reduced state, it holds one copy per energy level open to transport. Clearly d is now the total number of channels we intend to include in our scheme; this quantity must be necessarily lower than the total number of spin-orbital modes, i.e. it must hold $d \leq 2M$. Let us write in the second quantization language, the more general pure conduction state which satisfies these requirements.

$$|BB_{out}\rangle = \prod_{E} \left(\sum_{\alpha,\beta}^{d} \Phi_{\alpha\beta} \;\; \hat{a}^{\dagger}_{1,\alpha}(E) \;\; \hat{a}^{\dagger}_{2,\beta}(E) \right) |0\rangle \; . \tag{3.1}$$

We required the state to be *pure* for simplicity of calculation, this leads to no loss of generality. In this expression, thanks to the first requirement we have only to sum over term containing exactly two construction operators, one for to each port. The second requirement simply tells us that the matrix $\Phi_{\alpha\beta}$ is not a function of E.

Let us write again the previous expression in a different notation that will recur often later in this work

$$|BB_{out}\rangle = \prod_{E} \hat{\mathcal{O}}^{\dagger}(E) \, |0\rangle \; ,$$

where

$$\hat{\mathcal{O}}(E) \equiv - \sum_{\alpha,\beta} \Phi^{\star}_{\alpha\beta} \, \hat{a}_{1,\alpha}(E) \, \hat{a}_{2,\beta}(E) \; . \tag{3.2}$$

These operators undergo particular commutation relations (descending from anticommutation relations for the $\hat{a}$ operators) which have been explicitly calculated in appendix C, since they will prove useful for future computation.

Note that the elements of the matrix $\Phi_{\alpha\beta}$ have a relevant physical meaning too, in fact if we write the state restricted to a single energy level, only channel degree of freedom remains and we obtain

$$|\Theta_{BB}\rangle = \sum_{\alpha,\beta}^{d} \Phi_{\alpha\beta} \, |\alpha\rangle_1 \otimes |\beta\rangle_2 \; .$$

So the entries of $\Phi_{\alpha\beta}$ are the components of $|\Theta_{BB}\rangle$ read in a separable orthogonal basis for the channel space.

Just keep in mind that with $|BB_{out}\rangle$ we are referring to the global many-body conduction state, while with $|\Theta_{BB}\rangle$ we are meaning the bipartite channel space state. They are two different levels of description which define the same thing; we shall call them respectively the *global* and *channel* black box states.

As a concluding remark, we would like to point out that the following equation is the normalization condition for both $|BB_{out}\rangle$ and $|\Theta_{BB}\rangle$.

$$\sum_{\alpha\beta} |\Phi_{\alpha\beta}|^2 \; = \; 1 \; ,$$

see appendix A for further details on this calculation.

3.2 Total current correlators averages

So far we presented the system and the tools we want to use in order to apply the needed measurements. Now we have to build a theoretical setup project which let us to acquire the wanted information on the state. First of all, as a naive approach to this problem, we want to show that assembling a satisfying setup is absolutely not a trivial task.

Let us simply estimate the total current which flows through a single port outcoming from the entangler, and the total current correlator between ports 1 and 2. By saying *total*, we mean that we are taking into account the contribution of all the channels, equivalently we may speak about non-channel-selective current. Our aim is to show that this very simple setup does not allow us to acquire any information on the channel black box state and therefore on entanglement neither.

Total current expectation

Let us choose for instance $j = 1$ (the result shall be exactly the same for $j = 2$).

$$\langle BB_{out}|\, \widetilde{I}_1(\omega)\, |BB_{out}\rangle = \frac{e}{h\nu} \sum_{E,\alpha} \langle BB_{out}|\, \hat{a}^\dagger_{1,\alpha}(E)\, \hat{a}_{1,\alpha}(E+\hbar\omega)\, |BB_{out}\rangle$$

Let us consider separately the cases $\omega \neq 0$ and $\omega = 0$. We now show that the first one gives no contribution at all.

$$\begin{aligned}\langle \widetilde{I}_1\rangle_{\underline{\omega\neq 0}} = \frac{e}{h\nu} \sum_{E,\alpha} \langle 0|\, \hat{\mathcal{O}}(E)\, \hat{\mathcal{O}}(E')\, \hat{a}^\dagger_{1,\alpha}(E)\, \hat{a}_{1,\alpha}(E')\, \hat{\mathcal{O}}^\dagger(E)\, \hat{\mathcal{O}}^\dagger(E')\, |0\rangle = \\ \frac{e}{h\nu} \sum_{E,\alpha} \langle 0|\, \left(\hat{\mathcal{O}}\, \hat{\mathcal{O}}^\dagger\, \hat{a}^\dagger_{1,\alpha}\right)_E \left(\hat{a}_{1,\alpha}\, \hat{\mathcal{O}}\, \hat{\mathcal{O}}^\dagger\right)_{E+\hbar\omega} |0\rangle = \\ \frac{e}{h\nu} \sum_{E,\alpha} \langle 0|\, \hat{a}^\dagger_{1,\alpha}(E)\, \hat{a}_{1,\alpha}(E+\hbar\omega)\, |0\rangle = 0.\end{aligned}$$

To obtain the previous equation we simply switched commuting operators, and used other elementary properties according to appendices A and C. We still have to calculate $\langle \widetilde{I}_1\rangle_{\underline{\omega=0}}$. To do this we just sum the single channel average intensities we reviewed explicitly in appendix B. In conclusion we obtain:

$$\boxed{\langle \widetilde{I}_1(\omega)\rangle = \delta_{\omega,0}\, \frac{eV}{h}} \tag{3.3}$$

We essentially obtained again Landauer's result, as we could expect. Though, we lost any explicit dependence on $\Phi_{\alpha\beta}$, which means that performing this measure gives us no information on the channel state, and is consequently useless.

Total noise current correlator expectation

We now want to compute the following expectation value:

$$\frac{1}{h\,\nu}\,\mathcal{S}_{1,2}(\omega,\omega') = \langle \delta\widetilde{I}_1(\omega)\,\delta\widetilde{I}_2(\omega')\rangle =$$
$$= \frac{e^2}{h^2\nu^2}\sum_{E,E',\alpha,\beta}\Big(\langle BB_{out}|\hat{a}^\dagger_{1,\alpha}(E)\;\hat{a}^\dagger_{1,\alpha}(E+\hbar\omega)\;\hat{a}^\dagger_{1,\alpha}(E')\;\hat{a}^\dagger_{1,\alpha}(E'+\hbar\omega')\;|BB_{out}\rangle +$$
$$-\langle BB_{out}|\hat{a}^\dagger_{1,\alpha}(E)\;\hat{a}^\dagger_{1,\alpha}(E+\hbar\omega)\;|BB_{out}\rangle\langle BB_{out}|\;\hat{a}^\dagger_{1,\alpha}(E')\;\hat{a}^\dagger_{1,\alpha}(E'+\hbar\omega')\;|BB_{out}\rangle\Big)$$

It is possible to demonstrate that this expression gives a null result as long as at least one between ω and ω' is nonzero. So the only potentially interesting contribution is given by the case $\omega = \omega' = 0$, the shot noise. Channel current shot noise correlators are reviewed in appendix B; we just have to sum over all channel pairs to get the total noise correlator.

$$\langle \delta\widetilde{I}_1\,\delta\widetilde{I}_2\rangle_{\underline{00}} = \frac{e^3V}{\nu h^2}\left(1-\sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2\right) = 0.$$

Here we got a relevant result. By measuring total currents correlators, without making any state transformation, it is not possible to acquire any information on the channel state or its entanglement. It is pointless.

3.3 The four-leaded device

What we just learnt is that applying an amperometer to the outcoming ports and measure the total current is definitely not enough to obtain any relevant information on the channel state. Though we do not want to give up considering currents as an important tool capable of helping us in the pursue of our aim. Then, we manage to resolve this trouble by passing through one of two possible ways. Either *we perform some relevant transformations on the global state*, or *we change the approach we use to measure currents.*

Transforming the state means operatively to apply nanostructured components on the path of the carriers belonging to the system state. We shall include the action of this transforming device in our model as a coherent scattering process, involving 4 leads. Leads 1 and 2, which are connected to the black box, are incoming ports to our scatterer, then we define two new leads, 3 and 4 which are the outcoming ports. We assume that the this particular transmission process allows no backscattering. Therefore, Perform current and noise correlation measurements on the leads 3 and 4 is basically performing a measurement of the transformed state. We will refer to this specific scheme, suitable for most general classes of setups, as *four-leaded device*; it is represented in figure 3.1.

We now want to show that if we apply any *local* transformation onto the black box global state and measure only *total* outcoming noise correlations, we still get no information on the state. By saying 'local' we mean that the scattering matrix which describes the transformation process allows direct interaction only between ports 1 and 3, 2 and 4.

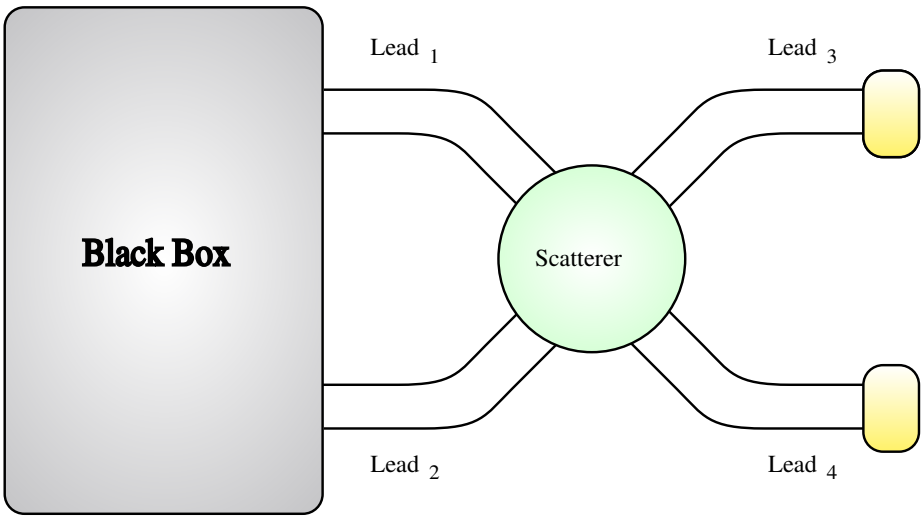

Figure 3.1: The four leaded device. Black box outcoming state, travelling through leads 1 and 2, undergo a state transformation given by a single scatterer, which allows no backscattering. The state is then gathered at contacts 3 and 4, and ready to be measured.

Then, let us write the form of the S-matrix. It is block diagonalized for different energy levels, since the scattering is elastic. Each block has dimension $2d \times 2d$ and locality condition tells us that its form must be the following

$$\left(\begin{array}{c} \hat{a}_{3,\alpha} \\ \hline \hat{a}_{4,\alpha} \end{array}\right) = \left(\begin{array}{c|c} S_{\alpha\beta} & 0 \\ \hline 0 & Z_{\alpha\beta} \end{array}\right) \left(\begin{array}{c} \hat{a}_{1,\beta} \\ \hline \hat{a}_{4,\beta} \end{array}\right) \tag{3.4}$$

Where S and Z are $d \times d$ unitary matrices; they could also depend on E, it does not influence the result. Now, we invert that relation, and write the black box state in terms of particle operators for leads 3 and 4.

$$\begin{aligned} |BB_{out}\rangle = \prod_E \left(\sum_{\alpha,\beta}^{d} \Phi_{\alpha\beta} \;\; \hat{a}^\dagger_{1,\alpha}(E) \;\; \hat{a}^\dagger_{2,\beta}(E) \right) |0\rangle = \\ = \prod_E \left(\sum_{\alpha,\beta,\rho,\sigma} \Phi_{\alpha\beta} \; S_{\rho\alpha} \; Z_{\sigma\beta} \;\; \hat{a}^\dagger_{3,\rho}(E) \;\; \hat{a}^\dagger_{4,\sigma}(E) \right) |0\rangle = \\ = \prod_E \left(\sum_{\rho,\sigma} \Psi_{\rho\sigma} \;\; \hat{a}^\dagger_{3,\rho}(E) \;\; \hat{a}^\dagger_{4,\sigma}(E) \right) |0\rangle \; . \end{aligned} \tag{3.5}$$

And we obtained a state which is again of the black box form, where $\Psi = (S \cdot \Phi \cdot Z^T)$. But we already showed that neither average total currents nor shot noise total current correlators can tell us anything on a black box state.

Another possible improvement of our setting, is the capability of measuring channel-selective currents and noise correlators. This simply means that we are able to measure separately any $\langle \delta I_{j,\alpha} \delta I_{k,\beta} \rangle$. This can be done if we manage

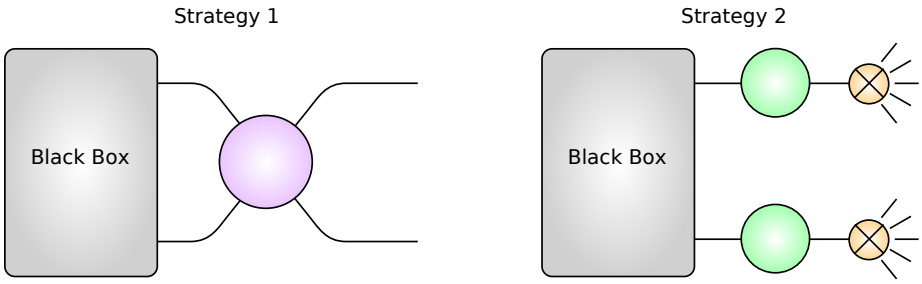

Figure 3.2: The two possible strategies. 1) we perform a non-local transformation which mixes the four leads all together. 2) we perform a local transformation which does not mix $1+3$ with $2+4$, but we have to include channel splitters in our setup.

somehow to split the flow of carriers belonging to different channels, into different paths, or leads. Actually, this is experimentally possible. If the channel degrees of freedom are given by spin, then it is possible to separate the channel contributions thanks to particular spin filters [11] [12]. If the channels are edge states in quantum Hall regime, then we use QPCs [2] [3] [4]. Any component able to separate channels is good for our purposes, we will refer to them as *channel splitters*. Measuring channel-selective currents and noise correlators recovers the possibility of getting useful information about the channel state even if we perform local transformations only.

In conclusion, we may choose between two possible types of successful strategies, also picted in fig. 3.2.

- *Strategy 1*: We perform non-local transformations and measure total outcoming noise correlators.
- *Strategy 2*: We perform only local transformations and measure channel-selective outcoming noise correlators.

Both these setup classes have been considered in recent works concerning this problem [2] [3] [4] [5] [6] [31], and remarkable results were achieved by choosing each one of them. We anticipate that in the present work we elaborate a setup for entanglement detection that follows the second strategy. Anyway, before explaining our proposal, we would like to review a couple of such protocols studied in the last years.

3.4 Electron Hong Ou Mandel Interferometer

This protocol was elaborated in 2006 by V. Giovannetti *et al* [6] on an idea proposed by G. Burkard *et al* [5] [32] [33] [34], its objective is to give a lower bound, via QEW, to the entanglement of the black box state by following the first entanglement detection strategy.

The main idea which supports this setup is the introduction of the electronic analog of the Hong-Ou-Mandel (HOM) optical interferometer [35]. In the optical physics context, this device can be seen like a beam splitter with transmission coefficient T, where two incoming complanar photon beams, directed at 45

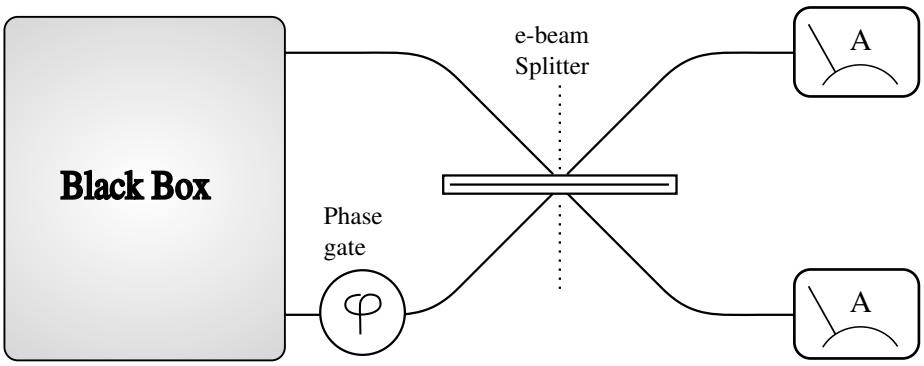

Figure 3.3: e-hom interferometer for entanglement detection, Schematic representation of its structural building. The black box state undergoes a local phase gate transformation and an e-hom interference; afterwards total current correlations are measured.

degrees form the normal vector, collide at the same point, like in figure 3.3. The S-matrix that describes our state transformation, decomposed in energy-diagonal $2d \times 2d$ blocks, appears as follows.

$$S(E) = \left(\begin{array}{c|c} \mathbb{I}\sqrt{1-T} & e^{i\varphi_\alpha}\sqrt{T} \\ \hline \mathbb{I}\sqrt{T} & -e^{i\varphi_\alpha}\sqrt{1-T} \end{array} \right) \qquad \forall\, E\,, \tag{3.6}$$

or equivalently,

$$\begin{cases} \hat{a}_{3,\alpha}(E) = \sqrt{1-T}\;\hat{a}_{1,\alpha}(E) + e^{i\varphi_\alpha}\sqrt{T}\;\hat{a}_{2,\alpha}(E) \\ \\ \hat{a}_{4,\alpha}(E) = \sqrt{T}\;\hat{a}_{1,\alpha}(E) - e^{i\varphi_\alpha}\sqrt{1-T}\;\hat{a}_{2,\alpha}(E) \end{cases} \tag{3.7}$$

We also included a channel-dependent phase gate transformation $e^{i\phi_\alpha}$, acting on lead 2 only: it is local transformation which acts before the e-HOM interference process.

The substantial feature of this electronic beam splitter is that it does not mix channels, as you can see from eqs. (3.6) (3.7). The protocol requires to perform a measure of the shot noise correlator between the total current intensities. We now explicitly calculate this outcome in order to understand which information on the state we acquire in this way.

$$\langle \delta\widetilde{I}_3\, \delta\widetilde{I}_4 \rangle_{\underline{00}} = \frac{e^2}{h^2\nu^2} \sum_{\alpha,\beta,E,E'} \Big(\langle\, \hat{a}^\dagger_{3,\alpha}(E)\; \hat{a}_{3,\alpha}(E)\; \hat{a}^\dagger_{4,\beta}(E')\; \hat{a}_{4,\beta}(E')\, \rangle + \\ - \langle\, \hat{a}^\dagger_{3,\alpha}(E)\; \hat{a}_{3,\alpha}(E)\, \rangle \langle\, \hat{a}^\dagger_{4,\beta}(E')\; \hat{a}_{4,\beta}(E')\, \rangle \Big)\ .$$

It is easy to see, by using simple commutation rules (see appendix C) that when $E \neq E'$ there is no contribution since the two summands cancels out. Then, only the term $E = E'$ remains, and thus the dependence on E of the particle operators can be omitted.

$$\langle \delta\widetilde{I}_3\, \delta\widetilde{I}_4\rangle_{\underline{00}} = \frac{e^2}{h^2\nu^2} \sum_{\alpha,\beta,E} \Big\{ \langle 0|\hat{\mathcal{O}} \left(\sqrt{1-T}\, \hat{a}^\dagger_{1,\alpha} + e^{-i\varphi_\alpha}\sqrt{T}\, \hat{a}^\dagger_{2,\alpha}\right) \times$$
$$\times \left(\sqrt{1-T}\, \hat{a}_{1,\alpha} + e^{+i\varphi_\alpha}\sqrt{T}\, \hat{a}_{2,\alpha}\right) \left(\sqrt{T}\, \hat{a}^\dagger_{1,\alpha} - e^{-i\varphi_\alpha}\sqrt{1-T}\, \hat{a}^\dagger_{2,\alpha}\right) \times$$
$$\times \left(\sqrt{T}\, \hat{a}_{1,\alpha} - e^{+i\varphi_\alpha}\sqrt{1-T}\, \hat{a}_{2,\alpha}\right) \hat{\mathcal{O}}^\dagger |0\rangle \Big\} - \frac{e^3 V}{h^2\nu}\ .$$

Let us separate the various contribution of the sixteen products in the expression. Each term is an expectation value of four $\hat{a}$ operators in a row. Only six of them assume a nonzero result, we list them by the port indexes of the four $\hat{a}$ operators.

$$\begin{aligned} \underline{1111} = \underline{2222} &\longrightarrow T(1-T) \\ \underline{1122} &\longrightarrow (1-T)^2 \\ \underline{2211} &\longrightarrow T^2 \\ \underline{1221} = \underline{2112} &\longrightarrow T(1-T)\left[w_\Phi - 1\right]\ . \end{aligned}$$

Let us sum all this terms in the equation. And we obtain:

$$\langle \delta\widetilde{I}_3\, \delta\widetilde{I}_4\rangle_{\underline{00}} = \frac{e^3 V}{h^2\nu} \left\{ 2T(1-T)\ +\ (1-T)^2\ +\ T^2\ + \right.$$
$$\left. +\ 2T(1-T)\left[w_\Phi - 1\right]\ -\ 1 \right\}\ ,$$

$$\boxed{\langle \delta\widetilde{I}_3\, \delta\widetilde{I}_4\rangle_{\underline{00}} = \frac{2e}{h\nu}\sqrt{\langle\widetilde{I}_3\rangle_{\underline{0}}\langle\widetilde{I}_4\rangle_{\underline{0}}}\ \times\ T(1-T)\left[w_\Phi - 1\right]\ .} \tag{3.8}$$

where by w_Φ is a real value, defined by the following expression.

$$w_\Phi \equiv \sum_{\alpha,\beta}\ \Phi^\star_{\alpha\beta}\ \Phi_{\beta\alpha}\ e^{i(\varphi_\alpha - \varphi_\beta)} \tag{3.9}$$

Let us investigate how this outcome is related to entanglement held by the black box state. Let us, for instance, assume that the local phase gate has no effect at all, i.e. $\varphi_\alpha = 0$. We now calculate over $|\Theta_{BB}\rangle$ the expectation value of $\mathbb{F}_d$, a quantitative entanglement witness of the Werner type.

$$\langle\Theta_{BB}|\mathbb{F}_d|\Theta_{BB}\rangle = \sum_{\alpha,\beta}\Phi^\star_{\alpha\beta}\langle\alpha\beta| \left(\sum_{j,k}|jk\rangle\langle kj|\right) \sum_{\rho,\sigma}\Phi_{\rho\sigma}|\rho\sigma\rangle =$$
$$= \sum_{\alpha,\beta}\Phi^\star_{\alpha\beta}\Phi_{\beta\alpha} = w_{\Phi,0} \tag{3.10}$$

where $w_{\Phi,0}$ is w_Φ without the phase gate transformation. We already acknowledged that an expectation value of a quantitative witness gives a lower bound on the degree of entanglement of the state. The conclusion is straightforward: just measuring once the shot noise current correlator we may ensure a lower bound for the entanglement of the channel state.

The purpose of adding to this discussion the phase gate is to generate further witnesses bu applying local channel transformations.

$$w_\Phi = \sum_{\alpha,\beta} \Phi^\star_{\alpha\beta} \langle \alpha\beta| \left(\sum_{j,k} e^{i(\varphi_j - \varphi_k)} |jk\rangle\langle kj| \right) \sum_{\rho,\sigma} \Phi_{\rho\sigma} |\rho\sigma\rangle = \langle \Theta_{BB} | \mathbb{F}_d^{(\varphi)} | \Theta_{BB} \rangle$$

And by construction $\mathbb{F}_d^{(\varphi)}$ is still a Werner QEW, since we can transform $\mathbb{F}_d$ into it by local unitary operations.

$$\left(e^{i\varphi_j} \otimes \mathbb{I}\right) \mathbb{F}_d \left(e^{i\varphi_j} \otimes \mathbb{I}\right)^\dagger = \mathbb{F}_d^{(\varphi)}$$

In other words, every time we run the experiment we may change the phase gate action on the state, and every time we get a new lower bound on the entanglement of $|\Theta_{BB}\rangle$.

Recently this work has even been extended to channel-entangled conduction states which do not satisfy the one-particle-per-energy-level requirement; and usefulness of fluffy-bunny entanglement has been discussed too. We refer to [31] for further reading.

3.5 Current correlators violation of Bell inequalities

Detecting a violation of Bell inequalities is another technique for experimental detection of entanglement; in fact that is an approach elaborated in earlier times than techniques based on entanglement witnesses.

The scheme we are about to review was proposed by C. Beenakker *et al* [2] [4] [8] [36], it appeared in the same article where they discussed the entanglement creation procedure for electric carriers in a Hall bar. Its objective is to detect entanglement, via Bell inequalities violation, in a doubly channeled (pseudospin) electron transport state by following the second entanglement detection strategy. Let us explain the theoretical idea which stands behind this type of setup.

The Clauser Horne Shimony Holt formulation [37] of the Bell inequalty is basically a test to distinguish quantum from classical correlation along different spin directions. If spin A is measured along unit vector $\underline{r}$, spin B is measured along $\underline{s}$, then the spin correlator is the expectation value

$$C_{rs} = \langle (\,\underline{r} \cdot \underline{\sigma}\,)_A \otimes (\,\underline{s} \cdot \underline{\sigma}\,)_B \rangle \ ,$$

where $\hat{\underline{\sigma}}$ are the Pauli matrices, with the standard choice of axis.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then the CHSH inequality may be written as follows.

$$\mathcal{B} \equiv |C_{rs} + C_{rs'} + C_{r's'} - C_{r's}| \leq 2 \tag{3.11}$$

for any choice of unit vectors $\underline{r}$, $\underline{r'}$, $\underline{s}$, and $\underline{s'}$. More precisely, if for any choice of these vectors $\mathcal{B} > 2$, then the state (which we took the expectation values

on) must be quantum entangled. Note that this is a sufficient condition only, since the inverse is not true: is is necessary for a pair of qubits to be strongly entangled to violate such relation.

An advantage of the CHSH form of the Bell inequalities generally is that it is still well defined when our experimental setting allows us to perform multiple but finite local spin measures only, rather than getting the exact expectation values. So, with $\langle N^A_{\pm}(\underline{r})\rangle$ we will mean the number of positive/negative outcomes when spin A is measured along $\underline{r}$. This is equivalent to assume our detection efficiency $\mathfrak{M}$ to be smaller than 1. In these terms, the inequalities reads:

$$\mathcal{B} = \frac{1}{\mathfrak{M}} \big|\langle (N^A_+(\underline{r}) - N^A_-(\underline{r}))(N^B_+(\underline{s}) - N^B_-(\underline{s}))\rangle + \langle (N^A_+(\underline{r}) - N^A_-(\underline{r}))\times \\ \times (N^B_+(\underline{s}') - N^B_-(\underline{s}'))\rangle + \langle (N^A_+(\underline{r}') - N^A_-(\underline{r}'))(N^B_+(\underline{s}') - N^B_-(\underline{s}'))\rangle - \\ \langle (N^A_+(\underline{r}') - N^A_-(\underline{r}'))(N^B_+(\underline{s}) - N^B_-(\underline{s}))\rangle \big| \leq 2 \, ,$$

where we set $\mathfrak{M} = \langle (N^A_+(\underline{r}) + N^A_-(\underline{r}))(N^B_+(\underline{s}) + N^B_-(\underline{s}))\rangle$; we reasonably assume that it is independent of the choice of vectors which we measure spins along. Now, in order to rewrite this expression in the shorter notation of (3.11), we may redefine spin correlators, this time taking into account imperfect efficiencies.

$$C_{rs} = \frac{\langle (N^A_+(\underline{r}) - N^A_-(\underline{r}))(N^B_+(\underline{s}) - N^B_-(\underline{s}))\rangle}{\langle (N^A_+(\underline{r}) + N^A_-(\underline{r}))(N^B_+(\underline{s}) + N^B_-(\underline{s}))\rangle} \, .$$

Equivalently, by setting $K_{\alpha\beta} = \langle N^A_\alpha(\underline{r}) N^B_\beta(\underline{s})\rangle$ (we are omitting its dependence from $\underline{r}$ and $\underline{s}$) we obtain:

$$C_{rs} = \frac{K_{\uparrow\uparrow} - K_{\uparrow\downarrow} - K_{\downarrow\uparrow} + K_{\downarrow\downarrow}}{K_{\uparrow\uparrow} - K_{\uparrow\downarrow} - K_{\downarrow\uparrow} + K_{\downarrow\downarrow}}$$

We still have to write the K_{ij} in terms of channel-selective current noise correlators.

$$K_{\alpha\beta} = \int_0^{\mathcal{T}_{det}} dt \int_0^{\mathcal{T}_{det}} dt' \, \langle \hat{I}_{A,\alpha}(t) \, I_{B,\beta}(t')\rangle = \tag{3.12}$$

$$= \mathcal{T}^2_{det} \, \langle \widetilde{I}_{A,\alpha}\rangle_{\underline{0}} \, \langle \widetilde{I}_{B,\beta}\rangle_{\underline{0}} \, + \int_{-\infty}^{+\infty} d\omega \, \frac{2\sin^2(\omega \mathcal{T}_{det}/2)}{\pi\omega^2} \times \\ \times \int_{-\infty}^{+\infty} dt \, e^{i\omega t} \, \langle \delta\hat{I}_{A,\alpha}(t) \, I_{B,\beta}(0)\rangle \, .$$

In conclusion, we wrote the CHSH-Bell inequalty in terms of average channel currents, and noise channel current correlators.

We have to be careful: in the expression (3.12) the channel current operators depend implicitly on the unit vector $\underline{r}$ we are measuring spin along. Precisely, $\hat{I}_{A,\alpha}(t)$ is the current of carriers in subsystem A which gave outcome α after a measurement of spin along $\underline{r}$ direction. This implies, operatively, that before we perform any channel current measures, we have to set the *spin filters* (or whatever device we use to separate channel components, e.g. QPCs if our

channel DOF is given by edge states in quantum Hall regime) to be splitting carriers along that specific spin directions.

As a consequence, we remark that in order to test the violation of the Bell inequalty on a single choice of four unit vectors (two per side), we will need to run the measurement process *four times*, one for each pair of chosen vectors, one for A and one for B. This concept shall be reviewed in detail in the next chapters.

Clearly, this filter direction setting operation can be equivalently seen as a unitary local transformation, and be included in our four-leaded device formulation mentioned previously.

In the next chapters we will describe the protocol we elaborated in this context, and give a detailed explanation on how it is supposed to work.

Chapter 4

Entanglement detection strategies II

We are about to present a theoretical scheme for measuring entanglement in a multi-channel conduction state, according to the standard coherent transport model in mesoscopic condensed matter systems. We will follow the second entanglement detection strategy (see sec. 3.3).

4.1 System requirements

We will suppose to have at our disposal a device, known as black box or entangler characterized by two outcoming spatially separated electrical ports, as in figure 4.1. This device prepares a potentially channel-entangled state, which we assume to be in the form (3.1). The entanglement detection involves current noise correlation measurements [5] [24] [32] [33], and it is conceptually based on the quantitative entanglement witnesses discussed in section 1.5. We need the measuring system to satisfy some requirements in order to be suitable for this purpose:

- It is possible to perform a certain set of *local* unitary operations, which will be defined later, on the black box state before measurements,
- It is possible to measure single channel currents according to a choice of an orthogonal basis for the channel DOF space (canonical channel basis),
- Several copies of the black box global state are at our disposal. Equivalently, the entangler is able to prepare the same state multiple times.

The aim of our analysis is to determine a set of lower bounds on the degree of entanglement of the channel state, in terms of both entanglement of formation and convex roof entanglement negativity. d will be the number of independent channel we are considering, i.e. the dimension of the channel DOF space.

To draw a model of this physical systems we will use the formulation of the four-leaded device (see sections 3.3). According to this picture, state transformation operations will be interpreted like coherent elastic scattering processes.

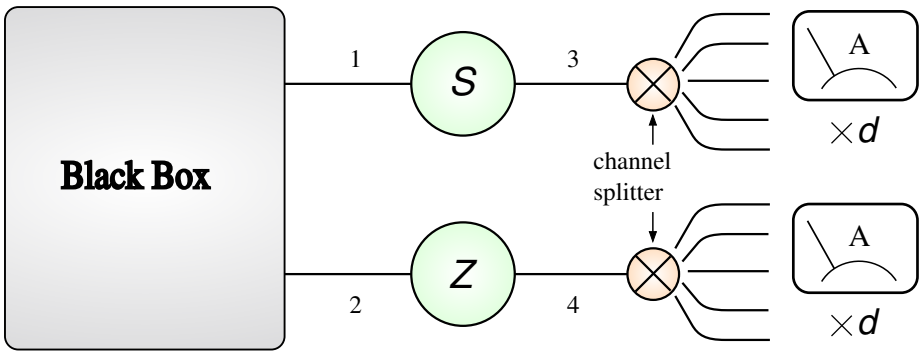

Figure 4.1: Scheme of the setup. The black box state undergoes the transformation given by two local gates, S and Z, then channel currents are splitted along the canonical basis, finally all the single channel current correlators are measured.

Moreover we will only take into account *local* transformation, characterized by $2d \times 2d$ energy-diagonal scattering matrices of the form (3.4), which yield the transformations

$$\begin{cases} \hat{a}_{3,\alpha}(E) = \sum_\beta \; S_{\alpha\beta}(E)\, \hat{a}_{1,\beta}(E) \\ \\ \hat{a}_{4,\alpha}(E) = \sum_\beta \; Z_{\alpha\beta}(E)\, \hat{a}_{2,\beta}(E)\ . \end{cases} \tag{4.1}$$

We also require that exists an orthonormal basis for the channel degree of freedom which we are able to split the particle flow along, via channel splitters. These splitters separate the electrons along a particular orthonormal separable basis for the channel degrees of freedom, which we will call *canonical* channel basis.

The detection scheme we are about to describe is consists in a sequence of *experimental runs*. According to the picture for the four-leaded device we described so far, a typical experimental run (or simply "run") is composed by three steps.

- *Step 1: setting of the local scattering devices.* We preset the scatterers so they will perform the local unitary transformation $S \times Z$. This is the only step that changes between consequent runs.

- *Step 2: black box activation.* We turn on the entangler; its global state is injected in the measuring device.

- *Step 3: channel currents recording.* We record simultaneously the $2d$ single channel currents and their crossterm noise correlators as functions of time.

The quantities of interest are indeed the zero frequency components of such signals, defined as

$$\langle \widetilde{I}_{3,\alpha} \rangle_{\underline{0}} = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_{-\mathcal{T}}^{+\mathcal{T}} dt\, \langle \hat{I}_{3,\alpha}(t) \rangle\ , \tag{4.2}$$

$$\langle \delta\widetilde{I}_{3,\alpha}\, \delta\widetilde{I}_{4,\beta}\rangle_{\underline{00}} = \lim_{\mathcal{T}\to\infty}\ \frac{1}{\mathcal{T}^2}\iint_{-\mathcal{T}}^{+\mathcal{T}} dt\, dt'\ \Big(\langle \hat{I}_{3,\alpha}(t)\, \hat{I}_{4,\alpha}(t')\rangle - \langle \hat{I}_{3,\alpha}(t)\rangle\, \langle \hat{I}_{4,\alpha}(t')\rangle,\Big)\ . \tag{4.3}$$

where $\mathcal{T}$ is the measurement time and $\hat{I}_{j,\alpha}(t)$ the current operator of eqs. (2.11) and (2.12). During any run, these expectation values will clearly depend both on the black box state *and* on the selected local transformation $S \otimes Z$.

4.2 Measurement results

We now discuss how much and which information on $\Phi_{\alpha\beta}$, and therefore on $|\Theta_{BB}\rangle$, we acquire after a run, as a function of S and Z. To do this we explicitly calculate the time averaged channel-selective currents and the channel shot noise current correlations.

Average channel currents

Let us compute the following expectation value for port 3

$$\langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}} = \frac{e}{h\nu}\sum_{E} \langle BB_{out}|\ \hat{a}^{\dagger}_{3,\alpha}(E)\ \hat{a}_{3,\alpha}(E)\ |BB_{out}\rangle\ . \tag{4.4}$$

Now, through eq. (4.1), we write the whole term as a function of particle operators for port 1 only, since the transformation, being local, forbids electron tunneling between leads 2 and 3. We also assume that $S(E)$ and $Z(E)$ do not depend on the energy level we are referring to: this assumption basically describes coherent centers of scattering in solid conductors when the energy window of modes open to transmission is sufficiently small. Under these approximations, eq. (4.4) gives

$$\langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}} = \frac{e}{h\nu}\sum_{E,\beta,\gamma} S^{\star}_{\alpha\beta}\ S_{\alpha\gamma}\ \langle 0|\ \hat{\mathcal{O}}(E)\ \hat{a}^{\dagger}_{1,\beta}(E)\ \hat{a}_{1,\gamma}(E)\ \hat{\mathcal{O}}^{\dagger}(E)\ |0\rangle\ . \tag{4.5}$$

The dependence on E vanishes, the sum can be replaced by the total number $\nu\, eV$ of discrete energy levels which participate to transmission. Exploiting the commutation relations of appendix C, we get in order to obtain

$$\begin{aligned}
\langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}} &= \frac{e^2V}{h}\sum_{\beta,\gamma} -S^{\star}_{\alpha\beta}\ S_{\alpha\gamma}\langle 0|\left[\hat{a}^{\dagger}_{1,\beta}\,,\,\hat{\mathcal{O}}\right]\ \left[\hat{a}_{1,\gamma}\,,\,\hat{\mathcal{O}}^{\dagger}\right]|0\rangle \\
&= \frac{e^2V}{h}\sum_{\beta,\gamma} S^{\star}_{\alpha\beta}\ S_{\alpha\gamma}\sum_{\rho,\sigma}\Phi^{\star}_{\beta\rho}\ \Phi_{\gamma\sigma}\ \langle 0|\ \hat{a}_{2,\rho}\ \hat{a}^{\dagger}_{2,\sigma}\ |0\rangle \\
&= \frac{e^2V}{h}\sum_{\beta,\gamma,\rho} S_{\alpha\beta}\ \Phi_{\beta\rho}\ (S_{\alpha\gamma}\ \Phi_{\gamma\rho})^{\star}\,.
\end{aligned} \tag{4.6}$$

The same calculation applies for port 4 as well, i.e.

$$\langle \widetilde{I}_{4,\alpha}\rangle_{\underline{0}} = \frac{e^2V}{h}\sum_{\beta,\gamma,\rho} Z_{\alpha\beta}\ \Phi_{\rho\beta}\ (Z_{\alpha\gamma}\ \Phi_{\rho\gamma})^{\star}\ . \tag{4.7}$$

That is the information we acquire by considering average channel currents. The above equations can be written in terms of expectation values of particular operators acting on the black box channel state $|\Theta_{BB}\rangle$. Indeed we have

$$\begin{cases} \langle \widetilde{I}_{3,\alpha} \rangle_{\underline{0}} = \dfrac{e^2 V}{h} \, \langle \Theta_{BB} | \left\{ \left(S^\dagger \, |\alpha\rangle\langle\alpha| \, S \right)_A \otimes \mathbb{I}_B \right\} |\Theta_{BB}\rangle \\ \\ \langle \widetilde{I}_{4,\alpha} \rangle_{\underline{0}} = \dfrac{e^2 V}{h} \, \langle \Theta_{BB} | \left\{ \mathbb{I}_A \otimes \left(Z^\dagger \, |\alpha\rangle\langle\alpha| \, Z \right)_B \right\} |\Theta_{BB}\rangle \, , \end{cases} \tag{4.8}$$

where now $|\alpha\rangle$ is the α-th vector of the channel canonical basis reduced to one subsystem only; S and Z are the same $d \times d$ unitary matrices of eq. (4.1).

This shows that by measuring average channel currents in a single run we are able to record separately the expectation values over $|\Theta_{BB}\rangle$ of all local operators of the form $(S^\dagger|\alpha\rangle\langle\alpha|S)_A \otimes \mathbb{I}_B$, and those who may be written as $\mathbb{I}_B \otimes (Z^\dagger|\alpha\rangle\langle\alpha|Z)_A$, for any vector $|\alpha\rangle$ of the reduced canonical channel basis.

Shot-noise channel current correlators

In general, according to our four-leaded device scheme, have two types of shot noise correlators: the *auto-correlation* $\langle \delta\widetilde{I}_{j,\alpha} \, \delta\widetilde{I}_{j,\beta} \rangle_{\underline{00}}$, and the cross-correlation $\langle \delta\widetilde{I}_{j,\alpha} \, \delta\widetilde{I}_{k,\beta} \rangle_{\underline{00}}$ for $j \neq k$ [24]. The information we are looking for will come our from the second type of terms, since entanglement is a measure of quantum non local correlations. Anyway, we calculate explicitly both of them.

- *Auto-correlator shot noise*

$$\langle \delta\widetilde{I}_{3,\alpha} \, \delta\widetilde{I}_{3,\beta} \rangle_{\underline{00}} = \frac{e^2}{h^2\nu^2} \sum_{E,E'} \langle BB_{out}| \, \hat{a}^\dagger_{3,\alpha}(E) \, \hat{a}_{3,\alpha}(E) \, \times \\ \times \left(\mathbb{I} - |BB_{out}\rangle\langle BB_{out}| \right) \hat{a}^\dagger_{3,\beta}(E') \, \hat{a}_{3,\beta}(E') \, |BB_{out}\rangle \, .$$

Terms for which $E \neq E'$ give no contribution, and clearly they are uncorrelated. Then only summands referring the same energy level remain, for instance

$$= \frac{e^3 V}{h^2 \nu} \sum_{\gamma,\eta,\rho,\sigma} S^\star_{\alpha\gamma} \, S_{\alpha\eta} \, S^\star_{\beta\rho} \, S_{\beta\sigma} \left(\langle 0| \, \hat{\mathcal{O}} \, \hat{a}^\dagger_{1,\gamma} \, \hat{a}_{1,\eta} \, \hat{a}^\dagger_{1,\rho} \, \hat{a}_{1,\sigma} \, \hat{\mathcal{O}}^\dagger \, |0\rangle + \right. \\ \left. - \langle 0| \, \hat{\mathcal{O}} \, \hat{a}^\dagger_{1,\gamma} \, \hat{a}_{1,\eta} \, \hat{\mathcal{O}}^\dagger \, |0\rangle \langle 0| \, \hat{\mathcal{O}} \, \hat{a}^\dagger_{1,\rho} \, \hat{a}_{1,\sigma} \, \hat{\mathcal{O}}^\dagger \, |0\rangle \right) \, .$$

Now we apply the usual commutation relations in order to obtain

$$= \frac{e^3 V}{h^2 \nu} \sum_{\gamma\eta\rho\sigma} S^\star_{\alpha\gamma} \; S_{\alpha\eta} \; S^\star_{\beta\rho} \; S_{\beta\sigma} \left(\delta_{\eta\gamma} \sum_{\zeta} \Phi^\star_{\gamma\zeta} \, \Phi_{\sigma\zeta} - \sum_{\xi,\theta} \Phi^\star_{\gamma\xi} \, \Phi_{\eta\xi} \, \Phi^\star_{\rho\theta} \, \Phi_{\sigma\theta} \right)$$

And then we may apply substitutions with channel current averages, we calculated little while ago.

$$\boxed{\langle \delta\widetilde{I}_{3,\alpha}\ \delta\widetilde{I}_{3,\beta}\rangle_{\underline{00}} = \delta_{\alpha\beta}\ \frac{e}{h\nu}\ \langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}} - \frac{1}{\nu\, eV}\ \langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}}\ \langle \widetilde{I}_{3,\beta}\rangle_{\underline{0}}} \tag{4.9}$$

We found that measuring the shot-noise correlators between channel currents belonging to the same port, we *do not receive any further information* than getting average channel currents themselves. This result appears to suit our intuition: it is exactly the same result we found in appendix B for shot-noise channel current auto-correlator for port 1. We could expect that the result should have been the same, this is due to the fact that after the local $S \otimes Z$ unitary transformation, the global state is still in the black box form, as we know from eq. (3.5). Hence, since the final expression do no depend directly on the channel black box state specifics, it had to be written that way.

- *Cross-correlator shot noise*

$$\langle \delta\widetilde{I}_{3,\alpha}\ \delta\widetilde{I}_{4,\beta}\rangle_{\underline{00}} = \frac{e^2}{h^2\nu^2} \sum_{E,E'} \langle BB_{out}|\ \hat{a}^\dagger_{3,\alpha}(E)\ \hat{a}_{3,\alpha}(E)\ \times$$
$$\times \Big(\mathbb{I} - |BB_{out}\rangle\langle BB_{out}|\Big)\ \hat{a}^\dagger_{4,\beta}(E')\ \hat{a}_{4,\beta}(E')\,|BB_{out}\rangle\ .$$

The path to solve this equation is very similar to the previous one. Again we discard any term in the sum of the type $E \neq E'$, since they give no contribution.

$$= \frac{e^3V}{h^2\nu} \sum_{\gamma,\eta,\rho,\sigma} S^\star_{\alpha\gamma}\ S_{\alpha\eta}\ Z^\star_{\beta\rho}\ Z_{\beta\sigma} \Big(\langle 0|\ \hat{\circlearrowleft}\ \hat{a}^\dagger_{1,\gamma}\ \hat{a}_{1,\eta}\ \hat{a}^\dagger_{2,\rho}\ \hat{a}_{2,\sigma}\ \hat{\circlearrowleft}^\dagger\ |0\rangle +$$
$$- \langle 0|\ \hat{\circlearrowleft}\ \hat{a}^\dagger_{1,\gamma}\ \hat{a}_{1,\eta}\ \hat{\circlearrowleft}^\dagger\ |0\rangle\langle 0|\ \hat{\circlearrowleft}\ \hat{a}^\dagger_{2,\rho}\ \hat{a}_{2,\sigma}\ \hat{\circlearrowleft}^\dagger\ |0\rangle\Big)\ .$$

Let us apply again commutation relations.

$$= \frac{e^3V}{h^2\nu} \sum_{\gamma,\eta,\rho,\sigma} S^\star_{\alpha\gamma}\ S_{\alpha\eta}\ Z^\star_{\beta\rho}\ Z_{\beta\sigma} \left(-\Phi^\star_{\gamma\rho}\Phi_{\eta\sigma} - \sum_{\xi,\theta} \Phi^\star_{\gamma\xi}\ \Phi_{\eta\xi}\ \Phi^\star_{\theta\rho}\ \Phi_{\theta\sigma}\right)$$

Now, while the second term in the expression may be written again as a function of the average channel currents, the first part does not: it is special. Let us write for instance the following proposition concerning the channel black box state.

$$\langle \alpha\beta|\ S \otimes Z\ |\Theta_{BB}\rangle = \langle \alpha\beta| \left(\sum_{p,q,r,s} S_{pr}\ Z_{qs}\ |pq\rangle\langle rs|\right) \sum_{cd} \Phi_{cd}|cd\rangle =$$
$$= \sum_{\eta\sigma} S_{\alpha\eta} Z_{\beta\sigma} \Phi_{\eta\sigma}$$

Once we acknowledged this property, our calculation leads immediately to the expression

$$\boxed{\langle \delta\widetilde{I}_{3,\alpha}\, \delta\widetilde{I}_{4,\beta}\rangle_{\underline{00}} \;=\; -\,\frac{1}{\nu\, eV}\,\langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}}\,\langle \delta\widetilde{I}_{4,\beta}\rangle_{\underline{0}} \;+\; \frac{e^3V}{h^2\nu}\,\Big|\langle \alpha\beta|\; S\otimes Z\;|\Theta_{BB}\rangle\Big|^2 .} \tag{4.10}$$

We found that considering shot-noise channel current correlators between ports 3 and 4 one gets the expectation values $|\Theta_{BB}\rangle$ of all operators of the form

$$(S\otimes Z)^\dagger\,|\alpha\beta\rangle\,\langle\alpha\beta|\,(S\otimes Z), \tag{4.11}$$

for any vector of the canonical channel basis $|\alpha\beta\rangle$. It is easy to see that, by just summing these terms over all β, for example, we obtain again the information achieved before via average channel currents. So including in the picture the shot noise correlators is really a *strict improvement* of the model.

We are now interested on how we can use this whole information to detect entanglement in the black box channel state $|\Theta_{BB}\rangle$.

Now that we acknowledged which information on $|\Theta_{BB}\rangle$ we can able to get from a run, our goal is to process this data to obtain expectation values of QEWs. This is the basic idea of our entanglement detection scheme. In the next chapter we explain precisely the complete execution of this scheme for a double channel system, i.e. $d = 2$. Then we will try to generalize that setup for any dimension of the channel degrees of freedom.

Chapter 5

Double-channel entanglement

We are now going to consider the particular case when the channel degeneracy to the transport states in the condensed matter conducting system is exactly 2. The channel degree of freedom space is now 2-dimensional, and this will be true along all this chapter. This is, for example, the case when we consider the electronic spin as the only channel degree of freedom.

Our actual issue is to decompose the standard QEW operators $\mathbb{G}_2$ and $\mathbb{F}_2$ as a combination of tensor product operators. The reason is simple: according to appendix D, it is impossible to evaluate QEWs in a single run, while it is possible for tensor product observables. Therefore, our idea is to perform various runs: after every run we evaluate a tensor product operator belonging to the decomposition of $\mathbb{G}_2$ or $\mathbb{F}_2$, and finally we calculate the expectation value of $\mathbb{G}_2$ or $\mathbb{F}_2$ themselves.

5.1 Pauli decomposition for standard QEWs

The Pauli matrices, i.e.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with the the identity form a basis of the operator space of a two-dimensional system Analogously, the tensor products between all ordered pairs of these operators $\sigma_j \otimes \sigma_k$ (identity included) form a *separable* basis for operators in the $\mathcal{H}_2 \otimes \mathcal{H}_2$ bipartite system.

The natural approach the QEW decomposition problem is to use this operator basis. Let us, for instance, write down explicitly in 4×4 matrix form our $\mathbb{G}_2$ and $\mathbb{F}_2$, defined in eqs. (1.6) and (1.8)

$$\mathbb{G}_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbb{F}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.1}$$

We now take these matrices and write them as combinations of tensor product between Pauli matrices. Here is the straightforward result

$$\begin{aligned} \mathbb{G}_2 &= \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right), \\ \mathbb{F}_2 &= \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right). \end{aligned} \tag{5.2}$$

We wrote these operators as a sum of four tensor product observables of the Pauli form, and one of them is the identity in both cases. Since the expectation value for identity is always 1, independently of the state we are considering, that particular term of the decomposition do not require any experimental run to be evaluated. Let us now focus of the other terms: in both cases they are all of the form $(\sigma_j \otimes \sigma_j)$ with $j \in \{x, y, z\}$. Every one of them is a single run observable but no pair containing two of them is simultaneous (see appendix D) since they do not locally commute,

$$[\sigma_j, \sigma_k] = 2i\, \varepsilon_{jkl}\, \sigma_l.$$

This consideration tells us that each single $(\sigma_j \otimes \sigma_j)$ operator requires a distinct experimental run to be evaluated, and we have three different of them. In conclusion, we may claim that with this particular choice we are capable of extracting the expectation values for both $\mathbb{F}_2$ and $\mathbb{G}_2$ exactly in *three runs*.

Alright, we achieved a relevant result; what we are allowed, and suggested, to do is to translate this choice of decomposing single run observables in terms of our four-leaded device language and respective entries of scattering matrices S and Z.

5.2 QEW equivalence for $d = 2$

We are now going to investigate if this particular decomposition *gives us further useful information* rather than obtaining only a pair of lower bounds for the entanglement of $|\Theta_{BB}\rangle$, respectively given by the expectation values of $\mathbb{F}_2$ and $\mathbb{G}_2$. So, for now, we concentrate on eventual hidden capabilities of the Pauli decomposition.

To this purpose, let us consider the following triplet local unitary transformations,

$$\sigma_j \otimes \mathbb{I} \qquad j \in \{x, y, z\}.$$

We just apply these U_{LOCC} transformations to the isotropic QEW $\mathbb{G}_2$ observable, in order to obtain other isotropic quantitative entanglement witnesses via the standard generation process. Then we write every one of them according to their respective Pauli decomposition. In conclusion, including $\mathbb{G}_2$, we obtain the following quadruplet of isotropic QEW.

$$\mathbb{G}_2^{\mathrm{I}} \equiv \mathbb{G}_2 = \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right)$$
$$\mathbb{G}_2^{\mathrm{II}} \equiv (\sigma_x \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_x \otimes \mathbb{I}) = \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$$
$$\mathbb{G}_2^{\mathrm{III}} \equiv (\sigma_y \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_y \otimes \mathbb{I}) = \frac{1}{2}\left(1 - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$$
$$\mathbb{G}_2^{\mathrm{IV}} \equiv (\sigma_z \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_z \otimes \mathbb{I}) = \frac{1}{2}\left(1 - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right)$$

We can apply these transformations for $\mathbb{F}_2$ observable as well, the result is very similar.

$$\mathbb{F}_2^{\mathrm{III}} \equiv \mathbb{F}_2 = \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) = \mathbb{I} - \mathbb{G}_2^{\mathrm{III}}$$
$$\mathbb{F}_2^{\mathrm{IV}} \equiv (\sigma_x \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_x \otimes \mathbb{I}) = \frac{1}{2}\left(1 + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right) = \mathbb{I} - \mathbb{G}_2^{\mathrm{IV}}$$
$$\mathbb{F}_2^{\mathrm{I}} \equiv (\sigma_y \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_y \otimes \mathbb{I}) = \frac{1}{2}\left(1 - \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right) = \mathbb{I} - \mathbb{G}_2^{\mathrm{I}}$$
$$\mathbb{F}_2^{\mathrm{II}} \equiv (\sigma_z \otimes \mathbb{I})\ \mathbb{G}_2\ (\sigma_z \otimes \mathbb{I}) = \frac{1}{2}\left(1 - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) = \mathbb{I} - \mathbb{G}_2^{\mathrm{II}}$$

There are two remarkable facts concerning these two quadruplets of observables. first of all, these eight QEWs may be written as a very simple combination of always the same single run observables: for instance the triplet $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$ and $\sigma_z \otimes \sigma_z$. But, in order to evaluate $\mathbb{G}_2$ and $\mathbb{F}_2$ we already got separately the expectation values of all these three tensor product observables. This implies that *without performing any further experimental run* we may get expectation values for all the written QEWs. And thus we have additional lower bounds on the degree of entanglement of $|\Theta_{BB}\rangle$ with no further efforts.

The second remark concerns the property that any $\mathbb{G}_2^{\Omega}$ operator (with Ω being I, II, III or IV) is equal to $\mathbb{F}_2^{\Omega}$, apart a sign and and adding an identity operator. Hence there is a simple linear correspondence between isotropic and Werner quantitative entanglement witnesses in the 2×2 dimensional space.

At first glance this should be an intriguing argument, because just by knowing one expectation value for an isotropic QEW, for example, we automatically know the outcome on the state for a Werner QEW (or vice versa). And thus we should have *two lower bounds* just by acquiring a single expectation value only, one Werner type bound and one isotropic.

Truthfully, this is not correct. In fact, the information that a pair QEWs (as long as they are connected by the relation $\widetilde{\mathbb{F}_2} = \mathbb{I} - \widetilde{\mathbb{G}_2}$) give us is exactly the same. To review this property, take the formulas we wrote in section 1.3 for the explicit values of $\mathcal{E}_{OF}$ and $\mathcal{E}_{CREN}$ calculated upon isotropic and Werner states. If we now restrict those results to the case $d = 2$ we obtain the following relations

$$\mathcal{E}_I(g) = \mathcal{E}_W(f = 1 - g) \qquad \text{and} \qquad \overline{\mathcal{E}_I}(g) = \overline{\mathcal{E}_W}(f = 1 - g)\ , \tag{5.3}$$

where g (resp. f) is the parameter which selects the specific isotropic (Werner) state, and it is exactly the expectation value over the same state for $\mathbb{G}_2$ ($\mathbb{F}_2$). Now, the QEW principle tells us that, for any state ρ

$$\begin{cases} \mathcal{E}_{OF}(\rho) \geq \mathcal{E}_I(Tr[\rho\,\widetilde{\mathbb{G}}]) \\ \\ \mathcal{E}_{OF}(\rho) \geq \mathcal{E}_W(Tr[\rho\,\widetilde{\mathbb{F}}]) = \mathcal{E}_I(1 - Tr[\rho\,\widetilde{\mathbb{F}}]) = \mathcal{E}_I(1 - Tr[\rho] + Tr[\rho\,\widetilde{\mathbb{G}}]). \end{cases}$$

In conclusion, isotropic and Werner type QEWs give the same information on the state.

Taking into account these arguments, we are allowed to say that when considering the case $d = 2$, then *isotropic and Werner QEW are equivalent*, under all aspects: they require the same amount of data to be evaluated, and at the same time they give the same information. No loss, no gain. This interesting property, which holds strictly in $\mathcal{H}_2 \otimes \mathcal{H}_2$ state space only, is a straightforward consequence that in two-qubits spaces the sets of isotropic and Werner states coincide perfectly. In conclusion, when studying this peculiar quantum space we can concentrate on working with either isotropic QEWs or Werner ones only, without loss of generality.

Let us take into account, for instance, quantitative entanglement witnesses of the isotropic type. Then, we will focus our attention on the previous quadruplet of isotropic QEWs ($\mathbb{G}_2^{\mathrm{I}}$, $\mathbb{G}_2^{\mathrm{II}}$, $\mathbb{G}_2^{\mathrm{III}}$ and $\mathbb{G}_2^{\mathrm{IV}}$); we have shown that they can be obtained either by adding or subtracting to each other the three single run operators $\sigma_j \otimes \sigma_j$. We are now supposed to write explicitly the local transformations which should be performed along each one of the three runs, and how we must assemble the current correlators shot noises to obtain the desired expectation values.

5.3 Scattering control

According to our picture, a total amount of three runs are needed. During each one of the we get the expectation value $\langle\Theta_{BB}|\sigma_j \otimes \sigma_j|\Theta_{BB}\rangle$ for a single j, either x, y or z. Every time we have to select determined S and Z matrices in order to obtain the right outcome. Now, remember the result achieved in the previous chapter relating shot noise channel current correlators to expectation values for operators acting on the channel DOF space; that is given by the equations (4.10) and (D.1). What that relation tells us is: we run the four-leaded device once, we gain all the distinct terms of the form

$$\langle\Theta_{BB}|\left[(S \otimes Z)^\dagger|\alpha\beta\rangle\langle\alpha\beta|(S \otimes Z)\right]|\Theta_{BB}\rangle$$

for any $\alpha, \beta \in 1, d$. In other words, current correlators give us exact prediction for the expectation values on $|\Theta_{BB}\rangle$ of a complete set of orthogonal one-dimensional projectors onto separable vectors. Then, according to this acknowledgement, we have simply to write the $\sigma_j \otimes \sigma_j$ operators as a combination of orthogonal 1-D separable projectors. Let us analyze the three cases.

- $\sigma_z \otimes \sigma_z$. This operator is already diagonal in the canonical channel basis; indeed it can be written as

$$\sigma_z \otimes \sigma_z = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| - |\downarrow\uparrow\rangle\langle\downarrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow|$$

and thus it does not need any state transformation in order to be evaluated. Here we write the recipe to extract its outcome starting from shot noise channel current correlators (and single average currents of course).

$$\begin{cases} S = Z = \mathbb{I} \\ \\ \dfrac{e^3 V}{\nu\, h^2}\ \langle \sigma_z \otimes \sigma_z \rangle \ = \ \displaystyle\sum_{\alpha,\beta}^{2} (-1)^{\alpha+\beta} \left[\langle \delta \widetilde{I}_{3,\alpha}\ \delta \widetilde{I}_{4,\beta} \rangle_{\underline{00}} + \frac{1}{\nu\, eV}\ \langle \widetilde{I}_{3,\alpha} \rangle_{\underline{0}} \langle \widetilde{I}_{4,\beta} \rangle_{\underline{0}} \right] \end{cases} \tag{5.4}$$

- $\sigma_x \otimes \sigma_x$. An immediate orthogonal separable basis for this observable is the set of all the possible tensor products of eigenstates of σ_x. Then we need, for both scatterers, a transformation which takes the eigenstates of σ_x into the eigenstates of σ_z (the canonical channel basis). Here we write the most general unitary matrix which satisfy this requirement.

$$X(p,q) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ip} & e^{ip} \\ e^{iq} & -e^{iq} \end{pmatrix} \qquad \text{for any } p, q \in \mathbb{R} \tag{5.5}$$

For our purposes, any choice of the phases p and q is acceptable. It is sufficient that any experimental setup which runs our protocol is able to perform *at least one* of the transformations belonging to this class; it does not matter which one of them, since any U_{LOCC} of the set does the trick. Indeed, it can be shown by simple matrices multiplication that $X^\dagger(p,q)\ \sigma_z\ X(p,q) = \sigma_x$, independently of the two real parameters. And, consequently

$$(X(p,q) \otimes X(r,s))^\dagger\ \ \sigma_z \otimes \sigma_z\ \ (X(p,q) \otimes X(r,s)) \ = \ \sigma_x \otimes \sigma_x$$

This equation tells us that, after transforming the state though $X \otimes X$, we calculate the expectation value of the involved observable $\sigma_x \otimes \sigma_x$ just like if we were calculating the outcome of $\sigma_z \otimes \sigma_z$ on the untouched state. And in conclusion the expectation value as a function of the channel current correlators, on ports 3 and 4, is exactly identical to that one we wrote before in equation (5.4)

$$\begin{cases} S = X(p,q)\ , \quad Z = X(r,s) \qquad \text{for any } \ p,q,r,s \in \mathbb{R} \\ \\ \dfrac{e^3 V}{\nu\, h^2}\ \langle \sigma_x \otimes \sigma_x \rangle \ = \ \text{as above} \end{cases}$$

- $\sigma_y \otimes \sigma_y$. This case is practically identical to the previous one, except for we have to write the most general 1-qubit space transformation which takes σ_y to σ_z. Here we write such unitary matrix.

$$Y(p,q) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ip} & -ie^{ip} \\ e^{iq} & ie^{iq} \end{pmatrix} \qquad \text{for any } p, q \in \mathbb{R} \tag{5.6}$$

And of course, once we transformed the channel state under $Y \otimes Y$, the

formula for the expectation value is equal to that one for $\sigma_z \otimes \sigma_z$.

$$\begin{cases} S = Y(p,q)\ , \quad Z = Y(r,s) \qquad \text{for any } \ p,q,r,s \in \mathbb{R} \\ \\ \dfrac{e^3 V}{\nu\, h^2}\ \langle \sigma_y \otimes \sigma_y \rangle \ = \ \text{as above} \end{cases}$$

This is what we need to do when operating on the four leaded device. In conclusion, the picture we just proposed requires that we are able to perform at least a transformation of the X type, and one of the Y type, this onto both the coherent wires outcoming from the entangler. It is interesting to see that *this is all the physical state manipulation we need* to make our protocol work out.

5.4 Protocol step by step for $d = 2$

After all this discussion, let us put all together the acknowledgements gained so far. We now write a brief review of the implementation of our protocol; focusing on operations, measurements and calculation needed to obtain four independent lower bounds on the degree of entanglement of the black box channel state $|\Theta_{BB}\rangle$.

- *Stage 1 - Physical apparatus triple running.* Execute three experimental runs for the four-leaded device, following steps as explained in section 4.1. Here we write the transformations which must be implemented along this measuring process.
  - First run: $S_1 = X(p,q)\ , \quad Z_1 = X(r,s)$
  - Second run: $S_2 = Y(p',q')\ , \quad Z_2 = Y(r',s')$
  - Third run: $S_3 = Z_3 = \mathbb{I}$

  Any real value for each one of the parameters is a good choice. Just select the parameters which mostly suits the experimental setting. For each run, obtain the 4 time-averaged channel currents $\langle \widetilde{I}_{3,\alpha} \rangle_{\underline{0}}$ and $\langle \widetilde{I}_{4,\alpha} \rangle_{\underline{0}}$, as well as the 4 shot noise channel current correlators between different ports $\langle \delta\widetilde{I}_{3,\alpha}\ \delta\widetilde{I}_{4,\beta} \rangle_{\underline{00}}$; for a total amount of data of 12 averages and 12 shot noise correlators.

- *Stage 2 - Expectation values for Pauli decomposition operators.* Compute the expectation values $\langle \Theta_{BB} | \sigma_j \otimes \sigma_j | \Theta_{BB} \rangle$ according to the following expression.

$$\begin{aligned} \langle \sigma_j \otimes \sigma_j \rangle = \Bigg[& \frac{h^2}{e^4\, V^2} \left(\langle \widetilde{I}_{3,\uparrow} \rangle_{\underline{0}} - \langle \widetilde{I}_{3,\downarrow} \rangle_{\underline{0}} \right) \left(\langle \widetilde{I}_{4,\uparrow} \rangle_{\underline{0}} - \langle \widetilde{I}_{4,\downarrow} \rangle_{\underline{0}} \right) + \\ & + \frac{\nu\, h^2}{e^3\, V} \Big(\langle \delta\widetilde{I}_{3,\uparrow}\ \delta\widetilde{I}_{4,\uparrow} \rangle_{\underline{00}} + \langle \delta\widetilde{I}_{3,\downarrow}\ \delta\widetilde{I}_{4,\downarrow} \rangle_{\underline{00}} - \langle \delta\widetilde{I}_{3,\uparrow}\ \delta\widetilde{I}_{4,\downarrow} \rangle_{\underline{00}} + \\ & \qquad - \langle \delta\widetilde{I}_{3,\downarrow}\ \delta\widetilde{I}_{4,\uparrow} \rangle_{\underline{00}} \Big) \Bigg]_{S_j \otimes Z_j} \end{aligned} \tag{5.7}$$

  where j stands for $\{x, y, z\}$. This data processing reduces all the information gathered so far in three real values only.

- *Stage 3 - Isotropic QEW quadruplet outcomes.* Keep in mind the Pauli decomposition, and calculate the expectation values for the following quadruplet of QEW observables of the isotropic type.

$$\begin{cases} g^{\mathrm{I}} = \langle\Theta_{BB}|\ \mathbb{G}_2^{\mathrm{I}}\ |\Theta_{BB}\rangle = \dfrac{1}{2}\Big(1 + \langle\sigma_x\otimes\sigma_x\rangle - \langle\sigma_x\otimes\sigma_x\rangle + \langle\sigma_x\otimes\sigma_x\rangle\Big) \\ g^{\mathrm{II}} = \langle\Theta_{BB}|\ \mathbb{G}_2^{\mathrm{II}}\ |\Theta_{BB}\rangle = \dfrac{1}{2}\Big(1 + \langle\sigma_x\otimes\sigma_x\rangle + \langle\sigma_x\otimes\sigma_x\rangle - \langle\sigma_x\otimes\sigma_x\rangle\Big) \\ g^{\mathrm{III}} = \langle\Theta_{BB}|\ \mathbb{G}_2^{\mathrm{III}}\ |\Theta_{BB}\rangle = \dfrac{1}{2}\Big(1 - \langle\sigma_x\otimes\sigma_x\rangle - \langle\sigma_x\otimes\sigma_x\rangle - \langle\sigma_x\otimes\sigma_x\rangle\Big) \\ g^{\mathrm{IV}} = \langle\Theta_{BB}|\ \mathbb{G}_2^{\mathrm{IV}}\ |\Theta_{BB}\rangle = \dfrac{1}{2}\Big(1 - \langle\sigma_x\otimes\sigma_x\rangle + \langle\sigma_x\otimes\sigma_x\rangle + \langle\sigma_x\otimes\sigma_x\rangle\Big) \end{cases}$$

  Of course, the user may consider a set of four Werner QEWs instead, in particular $\mathbb{F}_2^{\Omega} = \mathbb{I} - \mathbb{G}_2^{\Omega}$, for all Ω = I, II, III, or IV. Calculating their expectation values requires exactly the same efforts since $f^{\Omega} = 1 - g^{\Omega}$. But this would not influence the final result, since Werner and isotropic QEWs are equivalent for $d = 2$. At this point of the process it is evident whether entanglement has been detected or not. If any one of the outcomes g^{Λ} is greater than 1, then the protocol has detected entanglement in the state $|\Theta_{BB}\rangle$.

- *Stage 4 - Lower bounds on entanglement.* Take the subset of all the g^{Λ} which gave outcome greater than 1, and calculate the lower bound on the Entanglement of Formation of $|\Theta_{BB}\rangle$ fixed by the QEW principle, i.e.

$$\mathcal{E}_{OF}(\Theta_{BB}) \geq H_2\left(\frac{1 + \sqrt{g^{\Omega}\,(2 - g^{\Omega})}}{2}\right), \qquad \forall\, g^{\Omega} \geq 1$$

  with $H_2(x)$ defined by eq. (1.12). Here we have four independent lower bounds on the degree of entanglement of the black box channel state. The same argument can be used if we want to consider Convex roof entanglement negativity. In such case the bound should read:

$$\mathcal{E}_{CREN}(\Theta_{BB}) \geq \max_{\Omega}\left\{0\ ,\ \frac{g^{\Omega} - 1}{d - 1}\right\}$$

This completes our entanglement detection scheme.

5.5 Another interpretation of the isotropic QEW quadruplet

When we spoke for the first time about the set of isotropic states, we defined the operator $\mathbb{G}$ as an integer multiple of a projector onto a maximally entangled state. In the particular case we are actually considering $\mathbb{G}_2 = 2\,|\Phi^+\rangle\langle\Phi^+|$. Now, since the other QEWs of the quadruplet are obtained by applying a U_{LOCC} transformation on $\mathbb{G}_2$, they also must be one-dimensional projectors onto maximally entangled states (apart a scalar multiplication by 2). Precisely we have

$$\mathbb{G}_2^{\mathrm{II}} = (\sigma_x \otimes \mathbb{I})\, 2\left|\Phi^+\right\rangle\left\langle\Phi^+\right|(\sigma_x \otimes \mathbb{I}) = 2\left|\Psi^+\right\rangle\left\langle\Psi^+\right|$$
$$\mathbb{G}_2^{\mathrm{III}} = (\sigma_y \otimes \mathbb{I})\, 2\left|\Phi^+\right\rangle\left\langle\Phi^+\right|(\sigma_x \otimes \mathbb{I}) = 2\left|\Psi^-\right\rangle\left\langle\Psi^-\right|$$
$$\mathbb{G}_2^{\mathrm{IV}} = (\sigma_z \otimes \mathbb{I})\, 2\left|\Phi^+\right\rangle\left\langle\Phi^+\right|(\sigma_x \otimes \mathbb{I}) = 2\left|\Phi^-\right\rangle\left\langle\Phi^-\right| \ ,$$

where the involved state vectors are commonly known as Bell states.

$$|\Phi^{\pm}\rangle = \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}} \ , \qquad |\Psi^{\pm}\rangle = \frac{|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}$$

These four Bell states are not only maximally entangled, but they are all orthogonal to each other as well. Therefore, since they are four, they form a complete set for the $\mathcal{H}_2 \otimes \mathcal{H}_2$ space. Consequently, we may write any given pure state of our two-qubit space (for example $|\Theta_{BB}\rangle$) as a generic superposition of these states,

$$|\Theta_{BB}\rangle = \chi_1\left|\Phi^+\right\rangle + \chi_2\left|\Psi^+\right\rangle + \chi_3\left|\Psi^-\right\rangle + \chi_4\left|\Phi^-\right\rangle$$

According to this picture, the g values acquire an immediate physical meaning, they are the doubled square moduli of the vector components of $|\Theta_{BB}\rangle$ read in the Bell basis.

$$g^{\mathrm{I}} = 2\left|\langle\Phi^+|\Theta_{BB}\rangle\right|^2 = 2\left|\chi_1\right|^2 \qquad g^{\mathrm{II}} = 2\left|\langle\Psi^+|\Theta_{BB}\rangle\right|^2 = 2\left|\chi_2\right|^2$$

$$g^{\mathrm{III}} = 2\left|\langle\Psi^-|\Theta_{BB}\rangle\right|^2 = 2\left|\chi_3\right|^2 \qquad g^{\mathrm{IV}} = 2\left|\langle\Phi^-|\Theta_{BB}\rangle\right|^2 = 2\left|\chi_4\right|^2$$

Our protocol is capable of detecting entanglement if at least one of the outcoming g values is greater than 1. Now, suppose that g^{I}, for example, satisfies this inequality; this means that $|\chi_1| > 1/\sqrt{2}$. But since $|\Theta_{BB}\rangle$ is normalized (see appendix A) all other χ_j must have moduli lower than $1/\sqrt{2}$, and consequently no further g values, except for the first one, appear to be detecting any entanglement.

The present interpretation let us see easily that there exist entangled states, (even maximally) which can not be detected by our protocol. Take as an example the following pure state:

$$|\mathrm{Und}_1\rangle = \frac{|\uparrow\uparrow\rangle + i\,|\downarrow\downarrow\rangle}{\sqrt{2}} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(\left|\Phi^+\right\rangle - i\left|\Phi^-\right\rangle\right) \tag{5.8}$$

It is a maximally entangled state. Though its vector components read in the Bell basis are all equal or lower than $1/\sqrt{2}$, and therefore the state is clearly undetectable by our protocol.

5.6 Confrontation with other entanglement detection protocols

We just showed that our detection scheme is imperfect, since there are entangled states which are undetectable according to our picture. Anyway, a good question is if this proposal for an entanglement measurement system works better or

worse than other already existing protocols. As an example we will involve in our confrontation those setups we reviewed in chapter 3.

The protocol based on the electron-Hong Ou Mandel interferometer (Giovannetti *et al*, see section 3.4 and reference [6]) also works upon the QEW principle. Indeed, their final result is that they manage to calculate $\langle\Theta_{BB}|\mathbb{F}_2|\Theta_{BB}\rangle$, the expectation value on the channel black box state of the standard Werner quantitative witness. By the discussed principle of equivalence between Werner and Isotropic states, valid for $d = 2$, the information on the entanglement of $|\Theta_{BB}\rangle$ would be the same if we were considering the operator $\mathbb{G}_2^{\mathrm{III}}$. But our new protocol *includes strictly* the lower bound fixed by that isotropic QEW. Consequently the results our proposal is capable to give are in general strictly stronger than the outcomes of the e-HOM protocol.

It is important not to forget that e-HOM setup needs only a single experimental run in order to complete its procedure, while for ours three runs are necessary; moreover we require the capability to split channel currents, although the e-HOM interferometer protocol does not. But, apart from that, the great advantage of our protocol is that local transformation are the only ones needed. Therefore, differently for the e-HOM, we can apply it even when transformation which mix both the entangler outcoming ports are physically impossible to perform (for example because there is a macroscopic spatial separation between them).

The entanglement detecting protocol operating via Bell's inequalities violation (Beenaker *et al*, see section 3.5 and refs. [2] [4]) takes into account local operation only as well. The protocol is dependent on a choice of two pairs unit vectors $\underline{r}$ and $\underline{r'}$, $\underline{s}$ and $\underline{s'}$, and requires a total amount of *four* runs to be completed; moreover the entire process it can be repeated ad libitum, every time choosing a new set of unit vectors. In order to compare this protocol with ours we select the unit vectors which appear in the traditional formulation of CHSH inequalty: $\underline{r} = z$, $\underline{r'} = x$ for subsystem A, while $\underline{s} = (x+z)/\sqrt{2}$, $\underline{s} = (x-z)/\sqrt{2}$ for B. Therefore, if we define the following local observables

$$L = \sigma_z \otimes \mathbb{I} \qquad R = \mathbb{I} \otimes \frac{\sigma_x + \sigma_z}{\sqrt{2}}$$

$$L' = \sigma_x \otimes \mathbb{I} \qquad R = \mathbb{I} \otimes \frac{\sigma_x - \sigma_z}{\sqrt{2}} ,$$

we detect entanglement if the following inequality holds

$$|\langle LR + L'R + L'R' - L'R\rangle| > 2.$$

But now we can rewrite write the expression inside the brackets in terms of the Pauli decomposition. Then we obtain

$$\text{ß} = \frac{1}{2}\left(LR + L'R + L'R' - L'R\right) = \frac{1}{\sqrt{2}}\left(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z\right) = \\ = \sqrt{2}\left(\left|\Phi^+\right\rangle\left\langle\Phi^+\right| - \left|\Psi^-\right\rangle\left\langle\Psi^-\right|\right).$$

Now, given a density matrix ρ, In order to satisfy the Bell inequality violation $|Tr(\rho\,\text{ß})| > 1$ we can have two possible cases.

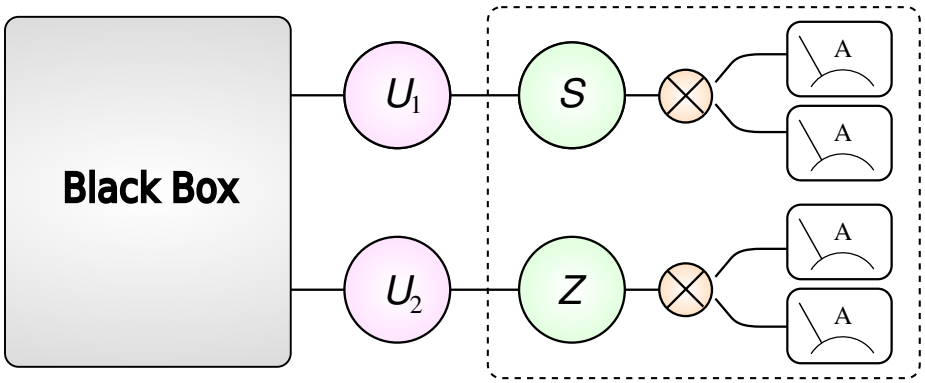

Figure 5.1: Upgrade setup. The local gates U_1 and U_2 act respectively before S and Z, and they must be constant along all the detection scheme.

- $Tr(\rho \, \text{Q}) > 1 \quad \longrightarrow \quad \sqrt{2} \langle \Phi^+ | \, \rho \, | \Phi^+ \rangle > 1 \quad \longrightarrow \quad Tr(\rho \, \mathbb{G}_2^{\mathrm{I}}) \geq \sqrt{2}$
- $Tr(\rho \, \text{Q}) < -1 \quad \longrightarrow \quad \sqrt{2} \langle \Psi^- | \, \rho \, | \Psi^- \rangle > 1 \quad \longrightarrow \quad Tr(\rho \, \mathbb{G}_2^{\mathrm{III}}) \geq \sqrt{2}$

In both cases, the expectation value on the state ρ for a certain $\mathbb{G}_2^{\Lambda}$ is greater than $\sqrt{2}$, but this means that our protocol detects entanglement for ρ too, since our detecting condition is that any g^{Λ} value overcomes 1.

In conclusion, we can claim that our protocol is stronger even than Beenakker's one, at least, when we make a single choice of unit vectors.

Truthfully, we may provide our protocol of some *adjustments* which may enlarge the range of detectable entangled states. Thanks to this upgrade it is straightforward to see that our QEW-based setup is definitely better than any procedure involving Bell's inequality violation. This adjustment consist basically in adding *constant local one-qubit gates* between the black box and our measuring device.

5.7 Upgrading the protocol via one-qubit gates

The entanglement detecting protocol we described in this chapter uses a quadruplet of isotropic QEW in order to establish four lower bounds on the degree of entanglement of the black box channel state. But what about all other isotropic quantitative witnesses we did not consider? Indeed, we took into account only four elements of the set

$$\mathrm{QEW}_{Iso} = \left\{ (U_1 \otimes U_2) \, \mathbb{G}_2 \, (U_1 \otimes U_2)^\dagger \mid \forall \, U_1, U_2 \text{ one qubit unitary matrices} \right\}.$$

If we find a way to get the expectation values over other observables belonging to this set, we could discover new lower bounds for the degree of entanglement of $|\Theta_{BB}\rangle$. Unfortunately, this is not possible without further efforts. The three runs included in our protocol let us to extract the expectation values of the three operators $\sigma_j \otimes \sigma_j$, which allow us to evaluate exactly four g values.

Instead, let us see what happens if we *repeat the entire procedure* but applying a local transformation onto the state before it undergoes all the operations scheduled in the protocol. We are talking about a pair of one qubit gates which we require to be *fixed and constant along all the three runs* we need to complete the procedure.

$$|\Theta'\rangle = (U_A \otimes U_B)\, |\Theta_{BB}\rangle$$

What we learned in this chapter is that, if we perform correctly the steps of our protocol, we are able to evaluate the outcomes of the quadruplet $\mathbb{G}_2^{\Omega}$ of observables on any transport state moving towards our measuring device. In other words, we get

$$\langle\Theta'|\, \mathbb{G}_2^{\Omega}\, |\Theta'\rangle = \langle\Theta_{BB}|\, \left[(U_A \otimes U_B)^{\dagger}\, \mathbb{G}_2^{\Omega}\, (A_1 \otimes U_B)\right] |\Theta_{BB}\rangle;.$$

This means that we extracted the expectation values on $|\Theta_{BB}\rangle$ of a *different* quadruplet of isotropic QEW. So we added other information, in terms of four further lower bounds, on the entanglement of the state. Figure 5.1 explains the idea of this upgrade implementation.

But doing this has a price. That is because we had to repeat the experimental part of the protocol as well, for a total of six runs in order to obtain eight lower bounds. In conclusion we can scan the entanglement of $|\Theta_{BB}\rangle$ using a larger amount of operators in the isotropic QEW set, but for every new quadruplet we must spend (at least) the same efforts of the first one.

This argument concludes our discussion on the double-mode case. In the next chapter we try to express a generalization for multi-channeled systems.

Chapter 6

Multi-mode entanglement

We now look for a generalization of the protocol we just sketched, which applied to the case $d = 2$. The main problem is that, while for qubit systems a deep knowledge was developed in the last decades, this is not the case with arbitrary finite dimensional quantum systems. Our actual purpose is to find a decomposition for the standard QEWs $\mathbb{F}_d$ and $\mathbb{G}_d$ as combinations of tensor product hermitian matrices. In the previous case we used the algebra of the Pauli matrices as a basis: there is not a conventional generalization of such algebra in higher dimension spaces. Then, we have to build a hand-made complete set of matrices for our decomposition.

It is also important to notice that, for $d > 2$, now Werner and isotropic sets are basically different: they are not isomorphic and their quantitative witnesses are not equivalent. Indeed, we do not receive the same information by considering one or the other class of QEW, and if we ant to take both of them into account we must treat them with two different manners.

6.1 Pairing decomposition for standard QEWs

During our research, we elaborated a particular decomposition for $\mathbb{G}_d$ and $\mathbb{F}_d$ which suits particularly well our purpose. Let us now define a basis of hermitian operators which act on a single subsystem, these will be the fundamental blocks which will compose our tensor product matrix set,

$$\begin{cases} \boxdot_j = |j\rangle\langle j| \\ \boxplus_{jk} = |j\rangle\langle k| + |k\rangle\langle j| \\ \boxminus_{jk} = i\,|j\rangle\langle k| - i\,|k\rangle\langle j| \end{cases} \qquad \text{for } j,k \in \{1..d\}, j \neq k.$$

In this picture, the pure states $|j\rangle$ are the vectors of the canonical channel basis. The set just presented has d^2 independent elements: they are d of the type $\boxdot$, $d(d-1)/2$ of the $\boxplus$ type and $\boxminus$ type as well. It is clear that the $\boxdot$ operators are one-dimensional projectors onto the canonical channel basis vectors, while the $\boxplus$ and $\boxminus$ matrices couple exactly two of them. For this reason we call the decomposition made according to these operators the *pairing decomposition*. To obtain a basis for the operators of the bipartite system we should consider the tensor products between all the ordered pairs of these matrices. It is easy

understand why we are interested in such decomposition. Consider the following equations:

$$\begin{aligned}\boxplus_{jk} \otimes \boxplus_{jk} &= |jk\rangle\langle kj| + |kj\rangle\langle jk| + |jj\rangle\langle kk| + |kk\rangle\langle jj| \\ \boxminus_{jk} \otimes \boxminus_{jk} &= |jk\rangle\langle kj| + |kj\rangle\langle jk| - |jj\rangle\langle kk| - |kk\rangle\langle jj|\end{aligned}$$

Just by adding or subtracting these two operators we obtain respectively $(|jk\rangle\langle kj| + |kj\rangle\langle jk|)$ or $(|jj\rangle\langle kk| + |kk\rangle\langle jj|)$, apart a factor 2. Recall now how we wrote the standard QEW observables. Since they are defined as $\mathbb{G}_d = \sum_{jk} |jj\rangle\langle kk|$ and $\mathbb{F}_d = \sum_{jk} |jk\rangle\langle kj|$, their pairing decomposition is straightforward.

$$\begin{cases} \mathbb{G}_d = \displaystyle\sum_{j=1}^{d} \boxdot_j \otimes \boxdot_j + \frac{1}{2} \sum_{k<j} \left(\boxplus_{jk} \otimes \boxplus_{jk} - \boxminus_{jk} \otimes \boxminus_{jk} \right) \\ \mathbb{F}_d = \displaystyle\sum_{j=1}^{d} \boxdot_j \otimes \boxdot_j + \frac{1}{2} \sum_{k<j} \left(\boxplus_{jk} \otimes \boxplus_{jk} + \boxminus_{jk} \otimes \boxminus_{jk} \right) \end{cases} \tag{6.1}$$

It is an encouraging result that both $\mathbb{G}_d$ and $\mathbb{F}_d$ are a combination of the same operators. At this level, we managed to write our standard QEWs into a decomposition which involves d^2 tensor product observables. This means that, if we want to use this particular decomposition, we will be able to extract the expectation values for $\mathbb{G}_d$ and $\mathbb{F}_d$ in *no more* than d^2 runs.

Later, we will do even better than that. But, for the moment, let us discuss if the chosen decomposition may bring us further advantages.

6.2 Combining pairing operators into QEWs

If we are really interested in adopting the pairing decomposition in order to evaluate the standard QEWs $\mathbb{G}_d$ and $\mathbb{F}_d$, a question which we should be interested in is the potential existence of other QEWs whose expectation values can be obtained by without further work on the experimental apparatus. In other words, we assume to have already completed all the runs and the operations on our measuring device in order to evaluate all the values of the form $\langle \boxdot_j \otimes \boxdot_j \rangle$, $\langle \boxplus_{jk} \otimes \boxplus_{jk} \rangle$ and $\langle \boxminus_{jk} \otimes \boxminus_{jk} \rangle$, for any $j \neq k$. Now, are we able to calculate the expectation values for other QEWs processing these data?

Well, the answer is yes, and we are now going to find them. Let Γ be any function defined upon the finite set $\{1..d\}$ whose allowed values are ± 1 only. We can build 2^d different Γ functions this way. Consider now the following unitary transformation acting on a single subsystem.

$$U(\Gamma) \equiv \sum_{k=1}^{d} \Gamma(k) \ |k\rangle\langle k| = U^{\dagger}(\Gamma).$$

This transformation is essentially a discrete phase gate, producing phase shifts additions which are multiples of π only. We now write how the pairing operators behave under such unitary operation.

$$\begin{cases} U(\Gamma)\ \boxdot_j\ U^\dagger(\Gamma)\ =\ \boxdot_j \\ U(\Gamma)\ \boxplus_{jk}\ U^\dagger(\Gamma)\ =\ \Gamma(j)\,\Gamma(k)\ \boxplus_{jk} \\ U(\Gamma)\ \boxminus_{jk}\ U^\dagger(\Gamma)\ =\ \Gamma(j)\,\Gamma(k)\ \boxminus_{jk}\,. \end{cases}$$

This is a very interesting result, every pairing operator is taken into itself apart a sign depending from Γ and the basis vector index. Now we apply to $\mathbb{G}_d$ and $\mathbb{F}_d$ the local transformation method to obtain further QEWs, in the following way:

$$\begin{aligned} \mathbb{G}_d(\Gamma) &= \Big(\mathbb{I}\otimes U(\Gamma)\Big)\,\mathbb{G}_d\,\Big(\mathbb{I}\otimes U^\dagger(\Gamma)\Big) \\ \mathbb{F}_d(\Gamma) &= \Big(\mathbb{I}\otimes U(\Gamma)\Big)\,\mathbb{F}_d\,\Big(\mathbb{I}\otimes U^\dagger(\Gamma)\Big)\,. \end{aligned}$$

Finding the pairing decomposition for such operators is now straightforward, since we simply need to apply the $\mathbb{I}\otimes U(\Gamma)$ transformation onto the respective decompositions for $\mathbb{G}_d$ and $\mathbb{F}_d$, i.e.

$$\begin{cases} \mathbb{G}_d(\Gamma) = \displaystyle\sum_{j=1}^{d} \boxdot_j\otimes\boxdot_j + \ \frac{1}{2}\sum_{k<j}\Gamma(j)\,\Gamma(k)\,\Big(\boxplus_{jk}\otimes\boxplus_{jk} - \boxminus_{jk}\otimes\boxminus_{jk}\Big) \\ \mathbb{F}_d(\Gamma) = \displaystyle\sum_{j=1}^{d} \boxdot_j\otimes\boxdot_j + \ \frac{1}{2}\sum_{k<j}\Gamma(j)\,\Gamma(k)\,\Big(\boxplus_{jk}\otimes\boxplus_{jk} + \boxminus_{jk}\otimes\boxminus_{jk}\Big)\,. \end{cases} \tag{6.2}$$

We found *a lot* of QEWs, both of the isotropic and the Werner type, whose decomposition involve exactly the same pairing operators as $\mathbb{G}_d$ and $\mathbb{F}_d$. This means many further lower bounds on the degree of entanglement held by $|\Theta_{BB}\rangle$ without any other experimental effort.

Let us investigate on how many different QEWs we have found this way. Now, if for any two Γ functions it held $\mathbb{G}_d(\Gamma)\neq\mathbb{G}_d(\Gamma')$ then this procedure allows us to build many different isotropic QEWs as many different two valued function on a set of d elements there are. Yet, it can be shown that $\mathbb{G}_d(\Gamma)=\mathbb{G}_d(-\Gamma)$; but no other equality between the $\mathbb{G}_d(\Gamma)$ operators holds. Same argument for the Werner QEWs.

So, in conclusion, if we have at our disposal the expectation values of all the operators $\boxdot_j\otimes\boxdot_j$, and those of the form $\boxplus_{jk}\otimes\boxplus_{jk}$ and $\boxminus_{jk}\otimes\boxminus_{jk}$, we may calculate the outcomes on the black box channel state of $2^{(d-1)}$ isotropic QEWs and $2^{(d-1)}$ Werner ones; for a total of $\underline{2^d\text{ lower bounds}}$ on entanglement.

This is a remarkable result achieved by working in the pairing decomposition; we are studying a state space whose dimension grows like d^2, and we found a simple way to calculate a certain number of lower bound on the quantity of entanglement of $|\Theta_{BB}\rangle$ which grows exponentially as a function of d. And, since we need only to measure via current correlators the expectation values of d^2 single run observables only, the protocol will not need more that d^2 runs to be completed.

In the next sections we will try to improve this result by determining the minimum number of runs which are needed to extract all the information required.

6.3 Simultaneity of the pairing operators

It is now important to acknowledge which and how many observables, among those appearing in the pairing decompositions of the $\mathbb{G}_d(\Gamma)$ and $\mathbb{F}_d(\Gamma)$ QEWs, may be evaluated simultaneously during a single experimental run. Indeed, this is the first target to achieve if our purpose is to improve our detection scheme by lowering the total number of runs needed.

In appendix D we provided the proof for a general property about simultaneity of single run observables. In particular, we showed that if we are given two tensor product hermitian operators, $T_1 \otimes U_1$ and $T_2 \otimes U_2$, then their expectation values can be extracted during the same run if they share a separable basis of eigenvectors, or equivalently if $[T_1, T_2] = [U_1, U_2] = 0$.

Let us apply this argument to those pairing operators we are interested in, as they are all tensor product observables. Now, since we found a decomposition for our standard QEWs in terms of operators of the form $O \otimes O$ we have to check the commutation relations only once per pair of such observables. To this purpose, let us write explicitly the commutators between local pairing operators.

- $\left[\boxdot_j, \boxdot_k\right] = 0$
- $\left[\boxdot_j, \boxplus_{kl}\right] = i\left(\delta_{jk} + \delta_{jl}\right)\ \boxminus_{kl}$
- $\left[\boxdot_j, \boxminus_{kl}\right] = i\left(\delta_{jk} - \delta_{jl}\right)\ \boxplus_{kl}$

It is clear that $\boxdot$ operators, being orthogonal projectors, commute with each other. This consequently means, for example, that the entire set of operators $\{\boxdot_j \otimes \boxdot_j\}$ is simultaneous: in theory they could all be measured in the same single run. Instead, $\boxdot$ matrices commute with $\boxplus$ ones (or $\boxminus$ ones) only if they do not share any value among their indexes. Let us move on, and calculate the remaining commutators, i.e.

- $\left[\boxplus_{jk}, \boxplus_{lm}\right] = -i\left(\delta_{jl}\,\boxminus_{km} + \delta_{jm}\,\boxminus_{kl} + \delta_{kl}\,\boxminus_{jm} + \delta_{km}\boxminus_{jl}\right)$
- $\left[\boxminus_{jk}, \boxminus_{lm}\right] = i\left(\delta_{jm}\,\boxminus_{kl} + \delta_{kl}\,\boxminus_{jm} - \delta_{jl}\,\boxminus_{km} - \delta_{km}\boxminus_{jl}\right)$
- $\left[\boxplus_{jk}, \boxminus_{lm}\right] = i\left(\delta_{jl}\,\boxplus_{km} + \delta_{kl}\,\boxplus_{jm} - \delta_{jm}\,\boxplus_{kl} - \delta_{km}\boxplus_{jl}\right)$

In other words, these operators commute with each other, regardless of their type ($\boxplus$ or $\boxminus$), only if their *four index values are all different*. The only exception is when we commute an operator with itself, and the result is zero by definition.

Therefore, what we have learnt is that as long as two observables belonging to our decomposition share at least an index value, they can not be evaluated during the same run. Otherwise it exists a local unitary transformation, capable of diagonalizing both of them. It is time for us to investigate how to write this transformation.

6.4 Run selection and scattering control

In order to gather the information we need for our study on entanglement, we have to perform a certain numbers of runs of any suitable experimental apparatus. Before we make the system undergo any run, we must select those

decomposing pairing observables we intend to observe during that run, and consequently choose which transformation and measurements apply. We should repeat the run procedure until all the expectation values of the paring observables we are interested in have been calculated.

So, we need to understand how are we supposed to control the scattering and measure the current correlators once we have chosen the simultaneous subset of pairing operators we want to consider as target for that run. It is evident that both measurement and transformation depend on the elements in the chosen set. We now try to express this dependence, and consider separately the cases whether a given pairing observable we want to evaluate belongs to one of the three types ($\boxdot$, $\boxplus$ or $\boxminus$).

- Assume that we included in our simultaneous set the operator $\boxdot_\alpha \otimes \boxdot_\alpha$, for a certain choice of the channel index α. By definition it follows that

$$\langle \Theta_{BB}| \boxdot_\alpha \otimes \boxdot_\alpha |\Theta_{BB}\rangle = |\langle \alpha\alpha|\Theta_{BB}\rangle|^2 = |\Phi_{\alpha\alpha}|^2$$

  From last section we learnt that it is not possible that the same simultaneous set contains another pairing operator which involves the same channel α. Consequently, we simply require that the scattering matrices S and Z act like the identity on the α-th channel. For instance

$$\begin{cases} S|_\alpha = Z|_\alpha = 1 \\ \\ \langle \boxdot_\alpha \otimes \boxdot_\alpha \rangle \;=\; \dfrac{\nu\, h^2}{e^3 V}\, \langle \delta\widetilde{I}_{3,\alpha}\, \delta\widetilde{I}_{4,\alpha}\rangle_{\underline{00}} + \dfrac{h^2}{e^4 V^2}\, \langle \widetilde{I}_{3,\alpha}\rangle_{\underline{0}} \langle \widetilde{I}_{4,\alpha}\rangle_{\underline{0}} \end{cases} \tag{6.3}$$

- If we wanted to calculate the expectation value of $\boxplus_{\beta\gamma} \otimes \boxplus_{\beta\gamma}$, it would be extremely useful noticing that it is equivalent than evaluating the Pauli operator $\sigma_x \otimes \sigma_x$ once we restricted our multi-mode quantum system into the 2-states system given by channels β and γ. Now, since no further simultaneous operators may involve these two channels, we apply on them an X transformation, just as the qubit case, defined in (5.5)

$$\begin{cases} S|_{\beta\gamma} = X(p,q) \qquad\qquad Z|_{\beta\gamma} = X(r,s) \qquad \text{for any } \; p,q,r,s \in \mathbb{R} \\ \\ \langle \boxplus_{\beta\gamma} \otimes \boxplus_{\beta\gamma} \rangle = \; \dfrac{h^2}{e^4\, V^2} \left(\langle \widetilde{I}_{3,\beta}\rangle_{\underline{0}} - \langle \widetilde{I}_{3,\gamma}\rangle_{\underline{0}} \right) \left(\langle \widetilde{I}_{4,\beta}\rangle_{\underline{0}} - \langle \widetilde{I}_{4,\gamma}\rangle_{\underline{0}} \right) + \\ \qquad\qquad + \dfrac{\nu\, h^2}{e^3\, V} \Big[\langle \delta\widetilde{I}_{3,\beta}\, \delta\widetilde{I}_{4,\beta}\rangle_{\underline{00}} + \langle \delta\widetilde{I}_{3,\gamma}\, \delta\widetilde{I}_{4,\gamma}\rangle_{\underline{00}} + \\ \qquad\qquad\qquad - \langle \delta\widetilde{I}_{3,\beta}\, \delta\widetilde{I}_{4,\gamma}\rangle_{\underline{00}} - \langle \delta\widetilde{I}_{3,\gamma}\, \delta\widetilde{I}_{4,\beta}\rangle_{\underline{00}} \Big] \end{cases} \tag{6.4}$$

- The last possibility is when we want to evaluate $\boxminus_{\rho\xi} \otimes \boxminus_{\rho\xi}$. But this is quickly done, since we restrict ourselves to the space spanned by ρ and ξ channels only, and it is immediate to see that our pairing operators behaves just like $\sigma_y \otimes \sigma_y$ on such 2-states space. And consequently we set

$$\begin{cases} S|_{\rho\xi} = Y(p,q)\;, \quad Z|_{\rho\xi} = Y(r,s) \qquad \text{for any } \; p,q,r,s \in \mathbb{R} \\ \\ \langle \boxminus_{\rho\xi} \otimes \boxminus_{\rho\xi} \rangle \;=\; \text{as above} \end{cases} \tag{6.5}$$

where the Y unitary matrices are defined by equation (5.6)

In conclusion, after we selected the pairing operators we want to estimate during the next run, we set the scattering matrices S and Z, according to the following simple rules.

For each $\boxplus \otimes \boxplus$ observable apply an X operator upon the pair of channels they involve. For each observable of the form $\boxminus \otimes \boxminus$, let the channels they refer to, undergo a Y transformation. All other channels must remain untouched.

After that, in order to compute the expectation values over the selected pairing observables from the current correlators, it will be sufficient to process the obtained data as explained by the previous equations.

6.5 Run minimization

Remember that, in order to complete the experimental stage scheduled in the protocol, we require that all expectation values $\langle \boxdot_\alpha \otimes \boxdot_\alpha \rangle$, $\langle \boxplus_{\alpha\beta} \otimes \boxplus_{\alpha\beta} \rangle$ and $\langle \boxminus_{\alpha\beta} \otimes \boxminus_{\alpha\beta} \rangle$ have been achieved. But, at the same time, we want to spend the least of the experimental efforts we could, in other words we are interested in performing the minimum number of runs.

Therefore, we have to divide all the operators belonging to our decomposition into a certain set of runs, being sure that we include them all, and within any run those which we select are all simultaneous. According to these rules, we want to find the partition algorithm which minimizes the numbers of run needed.

Local commutation rules for paring operators taught us that, during a single run, any vector of the canonical channel basis can not be targeted by more than one pairing observable at once. This principle suggest us to describe a run, by partitioning the canonical channel basis according to those selected pairing operators which act on them for that run.

Keeping this picture in mind, we formulated an equivalent problem which has the advantage of working with definitely more intuitive and visualizable objects. Once we managed to find a solution for this issue, we also have found the run minimization algorithm.

Equivalent minimization problem

Assume we have d numbered balls, and let all these balls be linked by strings. For each pair of balls it exist one and only one string which is bound to them both. In the beginning, all the balls and strings are *white*. We want to paint them, balls and strings, and we have a set of three colours at our disposal. Balls can be coloured by only one brush, say the *green* one; while we can paint the strings with *red* or *blue* colour. A single string may also be coloured by both red and blue brushes, in such case we will say that such string is *violet*.

All the painting is performed through a sequence of *rounds*. During a single round, we are allowed to use any number of brushes and to colour any number of objects; but we have to follow some rules. In fact, in the same round we are not allowed to:

- paint a ball and a string bound to it
- paint a pair of strings both bound to the same ball

- paint the same string of both red and blue colours.

We consider the entire painting operation completed as soon as all the balls are green, and all the strings are violet.

Our target is to find the algorithm which minimizes the number of rounds needed to complete the painting, as a function of the number of balls d.

6.6 Algorithm translation and protocol execution

We are now going to explain why and how the problem we just described is equivalent to our research on entanglement in the black box state based on channel current correlators measurements and QEW principles. The idea is very simple, all the objects painted in the n-th round are the pairing observables we measure during the n-th experimental run. Colouring in green the α-th ball is equivalent to get the expectation value of $\boxdot_\alpha \otimes \boxdot_\alpha$; while colouring the string which links the balls numbered β and γ with the red (resp. blue) brush, means to evaluate the operator $\boxplus_{\beta\gamma} \otimes \boxplus_{\beta\gamma}$ ($\boxminus_{\beta\gamma} \otimes \boxminus_{\beta\gamma}$). Therefore, requesting the whole graph be completely painted in the end, is basically asking that we got all the needed information in order to complete the runs. The painting rules basically forbid us to measure two non simultaneous observables in the same run.

This is essentially the map that translates a round of paintings into transformations and measures which have to be applied during a run. So let us now summarize the operations and calculations we have to perform to obtain the 2^d lower bounds on the entanglement of the black box channel state. We are now presenting the body of our protocol, which suits any given finite dimension d of the channel state space.

- *Stage 1 - Elaborate a painting algorithm.* As first step, find a proposal to solve the painting process. It is not important that this solution be the optimal one, which minimizes the number of rounds, as long as it completes the painting procedure.

- *Stage 2 - Translate and run the experiment.* By using the map we just provided, organize the runs which must be performed. For each run, set scatterers, measure channel current correlators and calculate expectation values according to which simultaneous pairing observables have been selected. The instructions to do this are listed in equations (6.3), (6.4) and (6.5).

- *Stage 3 - QEWs expectation values.* We calculate all the 2^{d-1} g-values and the 2^{d-1} f-values starting from the outcomes of all pairing observables which decompose $\mathbb{G}_d(\Gamma)$ and $\mathbb{F}_d(\Gamma)$ matrices, we found during the previous step. For instance, for every ± 1 valued Γ function on $\{1..d\}$ we compute the following values. Entanglement is detected any $g(\Gamma)$ is greater than 1 or any $f(\Gamma)$ lower than 0.

$$\begin{cases} g(\Gamma) = \sum_{\alpha=1}^{d} \langle \boxdot_\alpha \otimes \boxdot_\alpha \rangle + \frac{1}{4} \sum_{\beta \neq \alpha} \Gamma(\alpha)\,\Gamma(\beta) \left(\langle \boxplus_{\alpha\beta} \otimes \boxplus_{\alpha\beta} \rangle - \langle \boxminus_{\alpha\beta} \otimes \boxminus_{\alpha\beta} \rangle \right) \\ f(\Gamma) = \sum_{\alpha=1}^{d} \langle \boxdot_\alpha \otimes \boxdot_\alpha \rangle + \frac{1}{4} \sum_{\beta \neq \alpha} \Gamma(\alpha)\,\Gamma(\beta) \left(\langle \boxplus_{\alpha\beta} \otimes \boxplus_{\alpha\beta} \rangle + \langle \boxminus_{\alpha\beta} \otimes \boxminus_{\alpha\beta} \rangle \right) \end{cases}$$

- *Stage 4 - Lower bounds on entanglement.* Each QEW expectation value which succeeded in detecting entanglement gives a non trivial lower bound on $\mathcal{E}_{OF}$ and $\mathcal{E}_{CREN}$ of the black box channel state $|\Theta_{BB}\rangle$. In conclusion,

$$\begin{aligned} &\mathcal{E}_{OF}(\Theta_{BB}) \geq \mathcal{E}_W(f(\Gamma)), && \mathcal{E}_{CREN}(\Theta_{BB}) \geq \overline{\mathcal{E}_W}(f(\Gamma)), \\ &\mathcal{E}_{OF}(\Theta_{BB}) \geq \mathcal{E}_I(g(\Gamma)), && \mathcal{E}_{CREN}(\Theta_{BB}) \geq \overline{\mathcal{E}_I}(g(\Gamma)), \end{aligned}$$

  where the functions on the right side are defined by equations from (1.11) to (1.15). These inequalities hold for any $g(\Gamma)$ and any $f(\Gamma)$ over all possible Γ functions. We fulfilled our objectives, and the protocol is therefore completed.

The scheme just reviewed is basically a synthesis of our whole work. We consider the results we achieved rather satisfying, since we managed to find a way to extract a lot of information on the degree of entanglement of our multi-mode electron transport state by starting from an amount of data (and experimental efforts) which is relatively small.

As a further, interesting remark that we would like to include in this thesis, we will show a solution for the minimization problem discussed previously. So, according to the graph painting language, we will present an algorithm that completes the colouring issue along with the proof of its optimality.

6.7 Optimal solution for the graph painting problem

In order to explain how our optimal algorithm works, it is necessary to consider separately two possible cases, whether d is an even or odd integer. The solution indeed depends on the parity of the independent channel number, so we will start by discussing the odd-numbered case [38] for first.

To begin with, place the $d = (2k-1)$ white balls in order to make them occupy the vertexes of a d sided regular polygon. The stings, consequently, are the sides and the diagonals of that polygon. For ease of speaking, let us number the balls clockwise, starting from an arbitrary one. During the first round we colour the k-th ball with the green brush; at the same time we paint in red the side between vertexes 1 and d, along with all the diagonals *parallel* to it. By doing this we coloured every string which links balls j and $2k-j$, for all j from 1 to $k-1$. It is easy to see that, after this configuration, there are no more elements which can be coloured in the same round.

From the second to the d-th rounds we repeat exactly the same painting configuration of the first round, but rotated in respect to the center of the polygon, as shown in figure 6.1. Every round we choose a different side to be coloured in red, and we also colour the parallel diagonals and the most distant

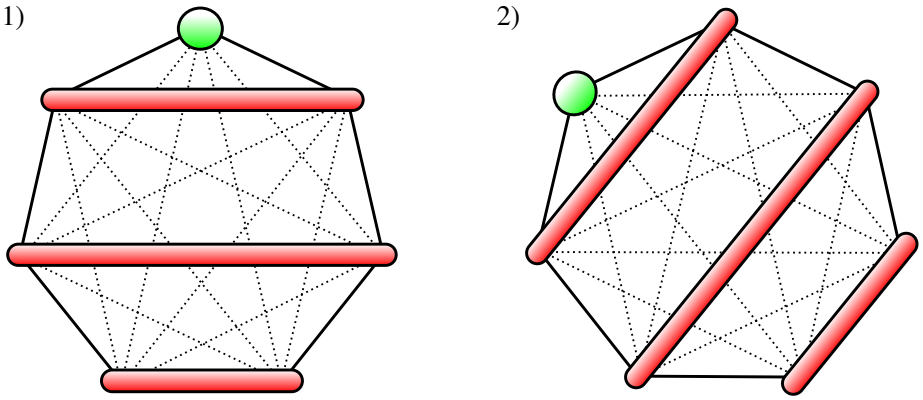

Figure 6.1: First two rounds of painting for the case $d = 7$. We begin by painting one side, the diagonals parallel to it, and the vertex which is left alone. Then we repeat the process but choosing another side, i.e. rotating the round painting configuration.

vertex. It is quite to see that, this way, we coloured all the balls in green and all the strings in red as well.

Now we repeat this process, but colouring the strings in blue this time (and of course there is no need to colour the balls once again). In conclusion we completed our painting process in $\underline{2d \text{ rounds}}$.

Let us now prove that this is the lowest number of rounds we can achieve. Since two adjacent strings can not be coloured simultaneously, during each round we can paint no more than $(d-1)/2 = k-1$ of them. But we have to paint twice (blue and red) a total amount of $d(d-1)/2$ strings. Then we have a lower bound on the round number any algorithm needs to its completition.

$$\sharp(\text{rounds}) \;\geq\; 2\,\frac{d(d-1)}{2}\,\left(\frac{d-1}{2}\right)^{-1} = 2d. \tag{6.6}$$

In conclusion, the minimizing algorithm we found is really the best we can do.

Let us approach the even numbered case, now. Place $d-1$ of the $d = 2k$ balls we have at the vertexes of an $d-1$ sided polygon, number them clockwise, and then place the d-th ball at the rotation center of such odd sided polygon. During the first round, we colour strings in red like we did in the odd $d-1$ case. But instead of colouring the k-th ball in green, we paint in red the string which bounds it to the one in the middle, just like in figure 6.2. Then we rotate this configuration until, at the end of round number $d-1$, all the strings have become red. Then do the same with the blue brush. Finally, during the last round, we paint all the balls in green. The painting is completed and we managed to to that in $\underline{2d-1 \text{ rounds}}$ total. Can we do better? The answer is no, because this procedure let us paint always the maximum number of strings per round for the first $2d-2$ rounds. But by doing so we are not allowed to paint any ball, so we need at least another round. This concludes the proof.

In conclusion we found a parity dependent algorithm which is the optimal solution for our graph painting problem. We complete it in

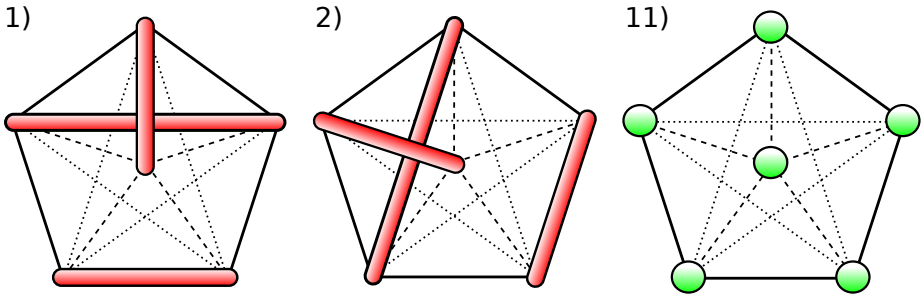

Figure 6.2: Case d=6. During the first round we paint in red one side of the pentagon, the diagonals parallel to it, and the string linking the vertex left alone with the ball in the center. Then we repeat this process choosing a different side, i.e. we rotate the painting configuration. When we are done with red brush, we do the same with blue one. During the eleventh round we paint all the balls.

- $2d$ rounds, for d odd
- $2d-1$ rounds, for d even

In both cases the number of rounds grows *linearly* with the channel space dimensions. This result is amazing: this protocol gives us a total amount of lower bounds exponential in d, after executing only about $2d$ runs of the experimental apparatus. This consideration also suggests us that probably our protocol works better as long as d grows; in other words, we are more likely capable of detecting multi-mode entanglement, if the physical system we are considering bears many channels open to transmission.

6.8 Further considerations

We want to point out some brief final remarks. First of all, we notice that it is possible to upgrade our protocol by using wisely local gates, just like we discussed for the double-channel case. The idea of this adjustment is exactly the same: we set some local transformation $(U_1 \otimes U_2)$ to act onto the black box global state before any $(S \otimes Z)$ protocol operation takes place. Thanks to these gates we will make our protocol capable of spacing over a larger set of QEW operators. Any time we repeat the protocol setting a new pair of local gates, at the usual cost of about $2d$ runs, we can acquire sufficient information to fix new 2^d lower bounds, half given by isotropic QEWs, half by Werner ones. We suggest to read section 5.7 for details concerning this technique, the generalization for multi-channel case is natural.

Moreover, it is important to remark that, even if along all this thesis we choose to work and operate on *pure* black box states, all the arguments which we presented can be in principle generalized *mixed* states as well.

Chapter 7

Conclusions

We studied entanglement in condensed matter systems, focusing on quantum correlations among electrons undergoing coherent conduction in mesoscopic structures. We were interested in particular internal degrees of freedom for electron carriers, which are independent of the transport dynamic, also known as channels. In our setting, channels represent the degree of freedom which may manifest entanglement properties. Our goal was to elaborate a theoretical operative scheme for channel entanglement detection and measurement in such transport systems.

We assumed to be provided of an unknown device (entangler or black box) whose role is to create conduction states potentially channel-entangled. We wanted to estimate its entangling capability by operating and measuring the outcoming two-terminal conduction state.

The scheme we formulated requires that we are able to operate onto the outcoming state by means of certain *local* unitary transformations, in particular those defined by eqs. (5.5) and (5.6). Onto the transformed state, we must perform *channel-selective current shot noise cross-correlation* measurements, as explained by eqs. (4.2) and (4.3).

The detection scheme we discussed, is conceptually based on *quantitative entanglement witness*. They are special physical observables, with the uncanny property that getting their expectation value over a state, automatically fixes a lower bound of the quantity of entanglement of that state. By considering two particular classes of states which show invariance under certain local symmetries, for instance isotropic and Werner sets of states, we managed to build a pair of non-equivalent standard QEWs, defined by eqs. (1.6) and (1.8), which work for any finite dimension of the channel space. We also elaborated an easy technique for generating further quantitative witnesses starting from these ones.

We succeeded in designing a scheme for entanglement detection, according to this picture, suitable for systems with any finite number d of channels involved: its execution program is dependent on d. We discovered that, if the steps of our scheme are executed correctly, it is possible to measure the expectation values on the black box state of $2^{(d-1)}$ distinct isotropic QEWs and $2^{(d-1)}$ Werner ones. This results in a total of 2^d lower bounds on the quantity of entanglement of the conduction state.

The scheme we elaborated needs that we repeat the whole process of state preparation + transformation + noise measurement, (which we usually call *run*),

more than once to reach its completition. We approached the problem of minimizing the total number of runs which allow us to evaluate the 2^d lower bounds; and we assembled an algorithm capable of reaching this target in $2d$ runs if d is and odd number, while $2d-1$ if d is even. We also demonstrated that it is not possible to improve this algorithm further on, therefore we actually solved successfully the minimization problem.

We showed that we can upgrade easily this scheme by introducing a constant local transformation to the state outcoming from the entangler. If we perform again the entire detection scheme after we set this adjustment, we are able to establish new 2^d lower bounds on entanglement, at the price of $2d$ (or $2d-1$) further runs. This scheme upgrade can clearly be repeated ad libitum.

In order to manipulate efficiently the elements of quantum mechanical theory which leaded us to these remarkable results, we set some general assumptions on the black box state (see section 3.1). An interesting proposal for further work in this context would be approaching the detection problem via QEW after weakening those assumptions. In other words, it would be very interesting to understand what can preserved of our project if we work on a larger space of possible conduction states.

Appendix A

Normalization of the black box state

Since our aim is to extract expectation values of certain observable on the black box states, it is necessary that we find their algebraic normalization. Firstly we will evaluate the condition on the matrix $\Phi_{\alpha\beta}$ which must be satisfied in order to normalize $|BB_{out}\rangle$. Let us rewrite the black box global state in the shorter form (using the $\hat{\mathcal{O}}$ set of operators).

$$|BB_{out}\rangle = \prod_E \hat{\mathcal{O}}^\dagger(E)\,|0\rangle\ ,$$

Now, let us calculate the term $\langle BB_{out}|BB_{out}\rangle$. Then we will require that this outcome is equal to 1.

$$\langle BB_{out}|BB_{out}\rangle = \langle 0|\prod_E \left(\hat{\mathcal{O}}(E)\right)\prod_{E'}\left(\hat{\mathcal{O}}^\dagger(E')\right)|0\rangle$$

In order to execute this calculation, we are suggested to help ourselves by using the commutation rules for the operator $\hat{\mathcal{O}}$, which have been explicitly reported in the appendix C. Now, since $\hat{\mathcal{O}}$ and its adjoint commute as long as they refer to different energies we may reorganize in some way the order of multiplication.

$$\langle BB_{out}|BB_{out}\rangle = \langle 0|\prod_E \left(\hat{\mathcal{O}}(E)\,\hat{\mathcal{O}}^\dagger(E)\right)|0\rangle$$

Then, we commute the pair $(\hat{\mathcal{O}}\,\hat{\mathcal{O}}^\dagger)$. This operation gives rise to four separate terms.

$$\langle BB_{out}|BB_{out}\rangle = \langle 0|\prod_E \left(\hat{\mathcal{O}}^\dagger(E)\,\hat{\mathcal{O}}(E) + \sum_\alpha \hat{?}_\alpha(E)\,\hat{a}_{1,\alpha}(E) + \right.$$
$$\left. + \sum_\alpha \hat{?}'_\alpha(E)\,\hat{a}_{2,\alpha}(E) + \sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2\right)|0\rangle$$

where $\hat{?}$ and $\hat{?}'$ are two expressions of second quantization operators that we do not need to write explicitly since they appear in terms which vanish anyway,

because $\hat{a}_{j,\alpha}\,|0\rangle = 0$. It is transparent that only the contribution of the fourth, and last, term remains. This is due to the fact that $|0\rangle$ belongs to the kernel of all the other summands. In other words we have:

$$\langle BB_{out}|BB_{out}\rangle = \langle 0|\prod_{E}\left(\sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2\right)|0\rangle = \Phi_{\alpha,\beta}\left(\sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2\right)^{\#E}, \quad \text{(A.1)}$$

where $\#E = \nu eV$ is the number of energy levels which participate to transmission.

It follows that a sufficient and necessary condition for the state $|BB_{out}\rangle$ to be normalized is

$$\boxed{\sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2 = 1\,.} \quad \text{(A.2)}$$

It is easy to see that such requirement automatically satisfies also the normalization for the state which describes only the channel degrees of freedom, for instance $|\Theta_{BB}\rangle$.

$$\langle\Theta_{BB}|\Theta_{BB}\rangle = \left(\sum_{\alpha,\beta}\Phi^{\star}_{\alpha\beta}\langle\alpha\beta|\right)\left(\sum_{\mu,\sigma}\Phi_{\mu\sigma}|\mu\sigma\rangle\right) = \sum_{\alpha,\beta}|\Phi_{\alpha\beta}|^2 = 1 \quad \text{(A.3)}$$

In conclusion, we showed that in order to normalize both the black box global and channel state we have to impose that the sum of the square moduli of all the elements of $\Phi_{\alpha\beta}$ is equal to 1. This proposition is assumed to be always true along the thesis.

One more remark: the normalization condition we just set, also ensures the following property to be satisfied.

$$\hat{\mathcal{O}}(E)\,\hat{\mathcal{O}}^{\dagger}(E)\,|0\rangle = |0\rangle \qquad \forall\, E \quad \text{(A.4)}$$

We will found the proposition written here quite useful in various situations.

Appendix B

Expectation values computation for channel current correlators

We now calculate explicitly the expectation values on the *untouched* global black box state $|BB_{out}\rangle$ (without applying any previous transformation), of two classes of operators: firstly we do it for the time-averaged channel current, then we apply the calculus to the (doubly time-averaged) noise channel current correlator. Remember that we suppose of being able to measure single-channel currents, since we may apply specific nanostructured devices (spin filters, QPCs) that physically split the particle flow into separate wires depending on the channel they belonged originally. As we said the following calculation is applied before making any state transformation, or equivalently j here can assume only values 1 or 2 (and neither 3 nor 4).

- *shot channel current.*

$$\langle \widetilde{I}_{j,\alpha} \rangle_{\underline{\omega=0}} = \frac{e}{h\nu} \sum_{E} \langle BB_{out}| \, \hat{a}^{\dagger}_{j,\alpha}(E) \, \hat{a}_{j,\alpha}(E) \, |BB_{out}\rangle$$

  The contribution of the $\hat{\mathcal{O}}$ operators related to energies different from E cancels out in the summatory.

$$\langle \widetilde{I}_{j,\alpha} \rangle_{\underline{\omega=0}} = \frac{e}{h\nu} \sum_{E} \langle 0| \, \hat{\mathcal{O}}(E) \, \hat{a}^{\dagger}_{j,\alpha}(E) \, \hat{a}_{j,\alpha}(E) \, \hat{\mathcal{O}}^{\dagger}(E) \, |0\rangle$$

  Through the commutation rules for $\hat{\mathcal{O}}$ which appear in appendix E we simply obtain the following result.

$$\boxed{\langle \widetilde{I}_{j,\alpha} \rangle_{\underline{0}} = \frac{e^2 V}{h} \left(\delta_{j,1} \sum_{\beta} |\Phi_{\alpha\beta}|^2 + \delta_{j,2} \sum_{\beta} |\Phi_{\beta\alpha}|^2 \right) .} \qquad \text{(B.1)}$$

  We also applied the substitution $\sum_E 1 = \nu e V$.

- *shot noise channel current correlators.* Note that with the word 'noise' we

simply mean expected variation from the mean value, for instance:

$$\langle \delta\widetilde{I}_{j,\alpha}\, \delta\widetilde{I}_{k,\beta}\rangle_{\underline{\omega=\omega'=0}} = \frac{e^2}{h^2\nu^2} \sum_{E,E'} \Big(\langle BB_{out}|\ \hat{a}^\dagger_{j,\alpha}(E)\ \hat{a}_{j,\alpha}(E) \times$$
$$\times\ \hat{a}^\dagger_{k,\beta}(E')\ \hat{a}_{k,\beta}(E') |BB_{out}\rangle - \langle BB_{out}|\ \hat{a}^\dagger_{j,\alpha}(E)\ \hat{a}_{j,\alpha}(E)\ |BB_{out}\rangle \times$$
$$\times \langle BB_{out}|\ \hat{a}^\dagger_{k,\beta}(E')\ \hat{a}_{k,\beta}(E')\ |BB_{out}\rangle \Big)\ .$$

It is easy to see that for every summand of the type $E \neq E'$ the two terms cancels out. Only terms $E = E'$ remain in the expression.

$$\langle \delta\widetilde{I}_{j,\alpha}\, \delta\widetilde{I}_{k,\beta}\rangle_{\underline{00}} = \frac{e^3 V}{h^2\nu} \Big(\langle 0|\ \hat{\mathcal{O}}\ \hat{a}^\dagger_{j,\alpha}\ \hat{a}_{j,\alpha}\ \hat{a}^\dagger_{k,\beta}\ \hat{a}_{k,\beta}\ \hat{\mathcal{O}}^\dagger\ |0\rangle +$$
$$- \langle 0|\ \hat{\mathcal{O}}\ \hat{a}^\dagger_{j,\alpha}\ \hat{a}_{j,\alpha}\ \hat{\mathcal{O}}^\dagger\ |0\rangle \langle 0|\ \hat{\mathcal{O}}\ \hat{a}^\dagger_{k,\beta}\ \hat{a}_{k,\beta}\ \hat{\mathcal{O}}^\dagger\ |0\rangle \Big)\ .$$

And again we simplify the equation thanks to the known commutation relations. It can be shown easily that, at this point, the result is different whether j and k are referring to the same port or not. Let us write down the ultimate result.

$$\boxed{\langle \delta\widetilde{I}_{1,\alpha}\, \delta\widetilde{I}_{1,\beta}\rangle_{\underline{00}} = \frac{e}{h\nu}\ \langle \widetilde{I}_{1,\alpha}\rangle_{\underline{0}} \left(\delta_{\alpha,\beta} - \frac{h}{e^2 V}\ \langle \widetilde{I}_{1,\beta}\rangle_{\underline{0}} \right)} \tag{B.2}$$

Idem for $j = 2$. in fact, while for $j = k$ the correlator is a function of the average currents only, when we consider the correlator between the two different ports ($j \neq k$) we can separate the explicit dependence on *every single entry* of $\Phi_{\alpha\beta}$.

$$\boxed{\langle \delta\widetilde{I}_{1,\alpha}\, \delta\widetilde{I}_{2,\beta}\rangle_{\underline{00}} = -\frac{1}{\nu e V}\ \langle \widetilde{I}_{1,\alpha}\rangle_{\underline{0}}\ \langle \widetilde{I}_{2,\beta}\rangle_{\underline{0}} + \frac{e^3 V}{\nu h^2}\ |\Phi_{\alpha\beta}|^2\ .} \tag{B.3}$$

As a consequence of this calculation, it is possible to measure the amplitudes of every entry of $\Phi_{\alpha\beta}$, just by measuring shot noise channel current correlators (though we lose any information on the phases).

Appendix C

Useful commutation relations

In this section some commutation relations involving $\hat{a}_{j,\alpha}(E)$ and $\hat{\mathcal{O}}(E)$ operators will be briefly reported. We simply write down the results, as the algebraic calculation made to obtain the following expressions is elementary and mechanic. Anyway, having these relations already computed and listed it may prove a useful tool to consistently speed up further calculations. So, we will present here these rules as a reminder for the reader.

For instance, let us recall that, according to our notation, $\{A, B\} = AB + BA$ is the anticommutator and $[A, B] = AB - BA$ is the commutator between two operators. Our starting point consists simply in the Fermi relations for the particles operators.

$$\left\{ \hat{a}_{j,\alpha}(E) \ , \ \hat{a}^{\dagger}_{k,\beta}(E') \right\} = \delta_{j,k} \ \delta_{\alpha,\beta} \ \delta_{E,E'} \ ,$$

where we have taken a discrete energy spectrum. Now we simply rewrite the definition of $\hat{\mathcal{O}}$.

$$\hat{\mathcal{O}}(E) \equiv - \sum_{\alpha,\beta} \Phi_{\alpha\beta} \ \hat{a}_{1,\alpha}(E) \ \hat{a}_{2,\beta}(E) \ .$$

Now, since it is a combination of terms all having exactly two particle operators, the previous anticommutation relation gives rise instead to commutation relations for $\hat{\mathcal{O}}$, as the minus signs cancel out every two exchanges of operators. Here we have those rules involving both $\hat{\mathcal{O}}$ and $\hat{a}$ operators together:

- $\left[\hat{a}^{\dagger}_{j,\alpha}(E), \hat{\mathcal{O}}(E') \right] = \delta_{E,E'} \sum_{\beta} \left(\delta_{j,2} \ \Phi^{\star}_{\beta\alpha} \ \hat{a}_{1,\beta}(E) - \delta_{j,1} \ \Phi^{\star}_{\alpha\beta} \ \hat{a}_{2,\beta}(E) \right) \ ,$
- $\left[\hat{a}_{j,\alpha}(E), \hat{\mathcal{O}}^{\dagger}(E') \right] = \delta_{E,E'} \sum_{\beta} \left(\delta_{j,1} \ \Phi_{\alpha\beta} \ \hat{a}^{\dagger}_{2,\beta}(E) - \delta_{j,2} \ \Phi_{\beta\alpha} \ \hat{a}^{\dagger}_{1,\beta}(E) \right) \ ,$
- $\left[\hat{a}_{j,\alpha}(E), \hat{\mathcal{O}}(E') \right] = \left[\hat{a}^{\dagger}_{j,\alpha}(E), \hat{\mathcal{O}}^{\dagger}(E') \right] = 0 \ .$

And then those involving $\hat{\mathcal{O}}$ alone:

- $\left[\hat{\mathcal{O}}(E), \hat{\mathcal{O}}^{\dagger}(E') \right] = \delta_{E,E'} \sum_{\alpha,\beta} \left\{ |\Phi_{\alpha,\beta}|^2 - \left(\sum_{\gamma} \Phi_{\alpha\gamma} \Phi^{\star}_{\beta\gamma} \right) \times \right.$

$$\left. \times \ \hat{a}^{\dagger}_{1,\alpha}(E) \ \hat{a}_{1,\beta}(E) - \left(\sum_{\gamma} \Phi_{\gamma\alpha} \Phi^{\star}_{\gamma\beta} \right) \hat{a}^{\dagger}_{2,\alpha}(E) \ \hat{a}_{2,\beta}(E) \right\} \ ,$$

- $\left[\hat{\mathcal{O}}(E), \hat{\mathcal{O}}(E')\right] = \left[\hat{\mathcal{O}}^\dagger(E), \hat{\mathcal{O}}^\dagger(E')\right] = 0$.

Appendix D

Single run observability and simultaneity

In this appendix, we discuss briefly the problem of which observables acting on channel space can be evaluated after performing a single run, according to the model presented in section 4.1. For brevity, we will call such operators *single run observables*. We will show that $\mathbb{G}_d$ and $\mathbb{F}_d$, the standard QEWs defined in eqs. (1.6) and (1.8), are not single run observables. Later, we will wonder when two or more single run observables can be evaluated in the same run.

Let us recall the information we may get after an experimental run, by taking into account average current measurement and cross-term shot noise channel current correlators. By eqs. (4.8) and (4.10) we obtained that the available data after a run is

$$\text{Run-Data}\,(S,Z) = \left\{ \left| \langle \alpha\beta |\; S \otimes Z \; |\Theta_{BB}\rangle \right|^2 \;;\; \forall \alpha,\beta \in \{1,d\} \right\}, \tag{D.1}$$

where S and Z are the selected local transformations for the run, and $|\alpha\beta\rangle$ are vectors of the canonical channel basis. Equivalently, we discovered that we get the expectation value of any one dimensional projector of the form

$$(S \otimes Z)^\dagger \, |\alpha\beta\rangle \, \langle \alpha\beta| \, (S \otimes Z).$$

But clearly, we also automatically have the expectation values of any operator $\mathrel{\underset{\frown}{=}}$ which can be written as a linear combination of this set of projectors. In other words it must hold

$$(S \otimes Z) \mathrel{\underset{\frown}{=}} (S \otimes Z)^\dagger = \sum_{\alpha,\beta} \mathcal{P}_{\alpha\beta} \, |\alpha\beta\rangle\langle\alpha\beta| \;, \tag{D.2}$$

This means that $\mathrel{\underset{\frown}{=}}$ can be diagonalized by means of local unitary matrices, i.e. it bears a separable orthonormal basis of eigenvectors.

Theorem. *An observable O on the channel degree of freedom state space is a single-run observable iff it exists a separable orthonormal basis of eigenvectors of O*

Observables acting on the channel space which do not satisfy this property, require more than one run to be evaluated.

We now demonstrate that neither $\mathbb{G}_d$ nor $\mathbb{F}_d$ satisfy this requirement.

We remember that $\mathbb{G}_d$ is an integer multiple of a 1-dimensional projector $\mathbb{G}_d = d|\Phi_d^+\rangle\langle\Phi_d^+|$; this means that any orthonormal basis of eigenvectors for $\mathbb{G}_d$ must necessarily include $|\Phi_d^+\rangle$, which is a maximally entangled state. For instance, $\mathbb{G}_d$ can not be a single run observable since it does not hold a separable basis of eigenvectors.

Let us now investigate upon $\mathbb{F}_d$. As we said previously, it has the function of swapping the state between subsystems A and B. So, suppose now that it exist a separable orthogonal eigenvector

$$\lambda_{\psi\phi}\ |\psi\rangle_A \otimes |\phi\rangle_B = \mathbb{F}_d\,(\,|\psi\rangle_A \otimes |\phi\rangle_B) = |\phi\rangle_A \otimes |\psi\rangle_B$$

then it must be $\lambda_{\phi\psi} = 1$ and $|\psi\rangle = |\phi\rangle$. So, in definitive, all potential separable eigenvectors of $\mathbb{F}$ are of the form $|w\rangle_A \otimes |w\rangle_B$. But in the $\mathcal{H}_d \otimes \mathcal{H}_d$ space, there are only d independent of these states, insufficient to be a complete set for a d^2 dimensional vector space. And thus, $\mathbb{F}_d$ neither is a single run observable.

We now take into account other isotropic or Werner quantitative entanglement witnesses, which are related to their respective standard QEW via U_{LOCC} operations (see section 1.6). Consider isotropic ones for example, and let $\mathbb{G'}_d = (U_1 \otimes U_2)\mathbb{G'}_d(U_1 \otimes U_2)^\dagger$. Since the composition of U_{LOCC} matrices is again a U_{LOCC}, if we now suppose $\mathbb{G'}_d$ be a single run observable

$$\sum_{\alpha\beta} \mathcal{P}_{\alpha\beta}|\alpha\beta\rangle\langle\alpha\beta| = (S \otimes Z)\ \mathbb{G'}_d\ (S \otimes Z)^\dagger =$$
$$= (S \otimes Z)(U_1 \otimes U_2)\ \mathbb{G}_d\ (U_1 \otimes U_2)^\dagger (S \otimes Z)^\dagger = (S' \otimes Z')\,\mathbb{G}_d\,(S' \otimes Z')^\dagger$$

then even $\mathbb{G}_d$ would, but we know it is not. The same argument holds for Werner QEWs as well. So, in conclusion, the present discussion leads to the following conclusion.

Theorem. *Quantitative entanglement witnesses are not single run observables. This property holds for both isotropic and Werner type QEWs.*

Tensor product observables

We now find a class of operators which are always single run observables. Consider the pair of observables O and O', each one acting on a single subsystem in the channel bipartite space. We write the global observable in the common a tensor product form $O_A \otimes O'_B$. From the spectral decomposition theorem, both O and O' are diagonalizable by a unitary transformation. Let now S and Z be the respective unitary matrices which diagonalize them. Then we have

$$(S \otimes Z)(O_A \otimes O'_B)(S \otimes Z)^\dagger = S\,O_A\,S^\dagger \otimes Z\,O_B\,Z^\dagger = D_A \otimes D'_B \qquad \text{(D.3)}$$

In other words, the transformed global observable is diagonal, and therefore $O_A \otimes O'_B$ is U_{LOCC} diagonalizable.

Theorem. *Tensor product observables $O_A \otimes O'_B$ are always single run observables.*

We say that a pair of single run observables is *simultaneous* if we can get both their expectation values in one run. From eq. (D.2) it is straightforward that two hermitian operators on the channel space C and D are simultaneous if they share a separable orthonormal basis of common eigenvectors. If this is true, there exist a choice for S and Z which diagonalize them both at the same time, and then they are both computable with the same run data.

Unfortunately, this can not easily translated in terms of commutators between operators. In fact, if $[O, O'] = 0$ these two operators share a common orthonormal basis of eigenvectors, but we do not know if it is exactly the basis of separable states.

Anyway this problem is quickly solved when we work on tensor product states. Consider in fact the pair of observables $O \otimes P$ and $O' \otimes P'$. Assume that $[O, O'] = [P, P'] = 0$. Now T and T' share a orthonormal basis of eigenvectors for subsystem A, and so do U and U' for B. We immediately obtain a separable orthonormal basis of eigenvectors for both $T \otimes U$ and $T' \otimes U'$. So they are simultaneous. This can be naturally generalized for any finite set of tensor product observables, and the result holds as long as every local operators commute with all the others.

Theorem. *Consider a finite set of tensor product observables $\mathcal{A} = \{T_j \otimes U_j\}$. If for all j and k it holds $[T_j, T_k] = [U_j, U_k] = 0$, then the whole $\mathcal{A}$ is a simultaneous set of single run observables.*

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